Estimation of pure qubits on circles

2007 J. Phys. A: Math. Theor. 40 1809

(http://iopscience.iop.org/1751-8121/40/8/009)

J. Phys. A: Math. Theor. **40** (2007) 1809–1834 doi:10.1088/1751-8113/40/8/009

Estimation of pure qubits on circles

Samuel L Braunstein[1], **Sibasish Ghosh**[1] **and Simone Severini**[2]

[1] Department of Computer Science, University of York, Heslington, York YO10 5DD, UK
[2] Department of Mathematics and Department of Computer Science, University of York, UK

E-mail: schmuel@cs.york.ac.uk, sibasish@cs.york.ac.uk and ss54@york.ac.uk

Received 11 December 2006
Published 6 February 2007
Online at stacks.iop.org/JPhysA/40/1809

Abstract

Gisin and Popescu (Gisin N and Popescu S 1999 *Phys. Rev. Lett.* **83** 432) have shown that more information about their direction can be obtained from a pair of anti-parallel spins compared to a pair of parallel spins, where the first member of the pair (which we call the pointer member) can point equally along any direction in the Bloch sphere. They argued that this was due to the difference in dimensionality spanned by these two alphabets of states. Here we consider similar alphabets, but with the first spin restricted to a fixed small circle of the Bloch sphere. In this case, the dimensionality spanned by the anti-parallel versus parallel alphabet is now equal. However, the anti-parallel alphabet is found to still contain more information in general. We generalize this to having N parallel spins and M anti-parallel spins. When the pointer member is restricted to a small circle these alphabets again span spaces of equal dimension, yet in general, more directional information can be found for sets with smaller $|N - M|$ for any fixed total number of spins. We find that the optimal POVMs for extracting directional information in these cases can always be expressed in terms of the Fourier basis. Our results show that dimensionality alone cannot explain the greater information content in anti-parallel combinations of spins compared to parallel combinations. In addition, we describe an LOCC protocol which extracts optimal directional information when the pointer member is restricted to a small circle and a pair of parallel spins are supplied.

PACS numbers: 03.65.Ta, 03.67.−a

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the quantum world there are many phenomena whose explanation is beyond the intuitions suggested by the classical world. For example impossibility of *cloning* [18] or *deleting* [15] an arbitrary quantum state, existence of *non-orthogonal* quantum states [16], impossibility of *spin flipping* [10], etc. Let us focus on this last concept. The spin degree of freedom of a spin $1/2$ system is described by a vector (or a mixture of projections on vectors) of a two-dimensional Hilbert space $\mathcal{H}$. The quantum mechanical operation which, when applied to a qubit $|\psi\rangle \in \mathcal{H}$, produces the qubit $|\psi^{\perp}\rangle \in \mathcal{H}$, orthogonal to $|\psi\rangle$, is called *spin flipping*. This operation exists if and only if the Bloch vector of $|\psi\rangle$ lies on *a given* great circle [8]. The classical analogue of spin flipping is the operation which, when applied to a vector (for example, in Eucledian space), produces its negative. This operation always exists.

It is provable that we can extract more information about the direction of the Bloch vector of $|\psi\rangle$ (for short, the *direction* of $|\psi\rangle$) if, instead of just $|\psi\rangle$, we are supplied with $|\psi\rangle \otimes |\psi\rangle$ or $|\psi\rangle \otimes |\psi^{\perp}\rangle$, in words, a pair of *parallel* or *anti-parallel* qubits. In a classical scenario, there is no difference between the parallel and the anti-parallel case, since the classical analogue of spin flipping always exists. Gisin and Popescu [10] have shown that, when concerning the direction of $|\psi\rangle$, anti-parallel qubits provide more information than parallel qubits. Note that if spin flipping were possible for each vector of $\mathcal{H}$, there would be no difference between these two cases. Gisin and Popescu pointed out that parallel qubits span a three-dimensional subspace of $\mathcal{H} \otimes \mathcal{H}$, while anti-parallel qubits span $\mathcal{H} \otimes \mathcal{H}$ entirely. Intuitively, vectors in an enlarged space are better distinguished than in the original space; the better we can distinguish the parallel (or the anti-parallel) qubits, the more information we can extract about the direction of $|\psi\rangle$. So, according to Gisin and Popescu, anti-parallel qubits contain more information about the direction of $|\psi\rangle$ compared to parallel qubits, since the former span a space of higher dimension.

The following question arises naturally: does this difference in extracting information about the direction of a qubit occur only for parallel and anti-parallel qubits? Can this be generalized to other cases? Let us illustrate the situation for general operations rather than just spin flipping. Consider an operation (not necessarily quantum mechanical) $\mathcal{A}$ on the pure state $|\psi\rangle$ of a spin $1/2$ system, such that $|\langle\psi|\mathcal{A}|\psi\rangle|$ is independent of $|\psi\rangle$. It can be shown that no non-trivial quantum mechanical operation satisfies this last requirement. In particular, spin flipping $\mathcal{A}$ should be of the form $\mathcal{A}|\psi\rangle = |\psi^{\perp}\rangle$, where $\langle\psi^{\perp}|\psi\rangle = 0$. One can raise now another question: which of the following two sets $\{|\psi\rangle \otimes \mathcal{A}|\psi\rangle : |\psi\rangle$ is any normalized qubit$\}$ and $\{|\psi\rangle \otimes \mathcal{B}|\psi\rangle : |\psi\rangle$ is any normalized qubit$\}$, where $\mathcal{A}$ and $\mathcal{B}$ are two operations such that both $|\langle\psi|\mathcal{A}|\psi\rangle|$ and $|\langle\psi|\mathcal{B}|\psi\rangle|$ are independent of $|\psi\rangle$, contains more information about the direction of the qubit? The problem can be posed in the following more general form. Let $f_i : \mathcal{H} \longrightarrow \mathcal{H}^{\otimes n}$, for $i = 1, 2, \ldots,$ be one-to-one maps which take $|\psi\rangle$ into some state of $\mathcal{H}^{\otimes n}$. Which f_i gives the largest amount of information about the direction of $|\psi\rangle$, when $|\psi\rangle$ is sampled according to some *a priori* probability distribution on $\mathcal{H}$? The maps f_i's are said to provide an *encoding* of vectors in $\mathcal{H}$. Note that spin flipping provides an encoding.

Bagan *et al* [2] have discussed this problem considering only those maps for which $f_i(|\psi\rangle)$ is an eigenstate of a total spin observable along some direction (specified by the direction of $|\psi\rangle$), when $|\psi\rangle$ is sampled from the uniform distribution on $\mathcal{H}$. According to their analysis, spin flipping does not play any special role; all that matters (in order to extract information) is the dimension of the subspace spanned by the states $f_i(|\psi\rangle)$.

In the present paper, we consider this *dimensional argument* in the context of estimating the direction of a pure qubit when the state $f_i(|\psi\rangle)$ is of the form $|\psi\rangle^{\otimes n} \otimes |\psi'\rangle^{\otimes m}$. Here the qubits $|\psi\rangle$ and $|\psi'\rangle$ are in one-to-one correspondence. Moreover we assume that the Bloch vectors of $|\psi\rangle$ and $|\psi'\rangle$ lie on two fixed different circles (possibly small). What is the motivation behind the choice of this encoding? First of all, small circles are a plausible first step generalization of great circles even though spin flipping does not exist for states from small circles. However, $|\psi\rangle \otimes |\psi\rangle$ and $|\psi\rangle \otimes |\psi^{\perp}\rangle$, where the Bloch vector of $|\psi\rangle$ lies on a given small circle, span three-dimensional subspaces. Thus, in our framework, the dimensional arguments of Gisin–Popescu and Bagan *et al* do not give any clue regarding best extraction directional information of the qubit. We find that, even in this case, anti-parallel qubits contain more information compared to parallel ones. More generally we see that the information contained in N qubits about the direction of $|\psi\rangle$, when $|\psi\rangle$ is encoded in the state $|\psi\rangle^{\otimes n} \otimes |\psi^{\perp}\rangle^{\otimes (N-n)}$ and taken with equal probability from a given small circle, decreases with the increment of the difference $|N - 2n|$.

We next consider the problem of estimating the direction of $|\psi\rangle$ with an encoding of the form $|\psi\rangle \otimes |\psi'\rangle$, where the Bloch vectors of $|\psi\rangle$ and $|\psi'\rangle$ lie respectively on two parallel circles S and S', and $|\psi'\rangle$ is in one-to-one correspondence with $|\psi\rangle$. Note that encoding parallel and anti-parallel qubits are special cases of this type of encoding. Given two circles S and S', we have an expression for $F(S, S')$, the maximum amount of information about the direction of $|\psi\rangle$. Given a circle S, one can calculate the maximum and minimum value of $F(S, S')$ over all possible choices of S'. Let us denote by $F^{\max}(S)$ and $F^{\min}(S)$ the maximum and the minimum value of $F(S, S')$, respectively. We show that $F^{\max}(S) \geqslant F(S, S^{\perp}) \geqslant F(S, S) \geqslant F^{\min}(S)$ for all S, where $S^{\perp}$ is the circle diametrically opposite to S. Also in this case the dimensional argument does not work. Another scenario in which the dimensional argument fails is the case of estimating the direction of a qubit sampled from a uniform distribution on set of two diametrically opposite circles. Here parallel and anti-parallel qubits provide same information about the direction of the qubit, even though they span spaces of different dimension.

Estimating a qubit from a circle is essentially estimating the phase of the qubit (see section 4 below and, e.g. [5]). Bearing this in mind, one can argue that the measurement basis used in the optimal estimation strategy should be the Fourier basis. This is also reflected by our results. In fact, we find that the measurement basis for the optimal strategy in the case of a qubit $|\psi\rangle$ uniformly distributed on a small circle, supplied the state $|\psi\rangle^{\otimes n} \otimes |\psi^{\perp}\rangle^{\otimes (N-n)}$ (where N is fixed), is the $(N+1)$-dimensional Fourier basis for every $n \in \{1, 2, \ldots, N-1\}$. This is also true if the supplied state is $|\psi\rangle^{\otimes n} \otimes |\psi'\rangle$, where $|\psi\rangle$ and $|\psi'\rangle$ are respectively from two different parallel circles and in one-to-one correspondence. This is expected as in the case of phase estimation. Moreover, we see that the measurement basis for the optimal strategy in the case of a qubit $|\psi\rangle$, uniformly distributed on a set of two diametrically opposite circles, supplied the state $|\psi\rangle \otimes |\psi\rangle$, is again the Fourier basis (in three dimensions). It is remarkable that this scenario does not correspond to phase estimation. Finally, we observe that, if the supplied state is $|\psi\rangle \otimes |\psi^{\perp}\rangle$ then the measurement basis of the optimal strategy is not the Fourier basis, but is in some way ”similar” to the Haar basis (see, e.g. [6] for this notion).

This paper is organized as follows. In section 2 we formulate the problem of state estimation discussed in the paper. In section 3 we sketch the related previous works. In section 4 we tackle the problem of estimating the direction of the Bloch vector of a qubit $|\psi\rangle$ sampled from a uniform distribution on a given small circle, when states of the form $|\psi\rangle^{\otimes n} \otimes |\psi^{\perp}\rangle^{\otimes m}$ are supplied. In section 5 we consider the case in which the supplied states are of the form $|\psi\rangle \otimes |\psi'\rangle$, where $|\psi\rangle$ and $|\psi'\rangle$ are respectively from two parallel circles and in one-to-one correspondence with each other. In section 6 we consider the case of a qubit $|\psi\rangle$ sampled from a uniform distribution on two diametrically opposite circles and the supplied states are either $|\psi\rangle \otimes |\psi\rangle$ or $|\psi\rangle \otimes |\psi^{\perp}\rangle$. In section 7 we describe an LOCC protocol for optimally estimating the direction of a qubit $|\psi(\theta, \phi)\rangle$ (θ is fixed), when a pair of parallel qubits is supplied. Section 8 is devoted to discussion and open problems.

2. The problem of state estimation

2.1. Formulation

Let us consider a quantum mechanical system with associated Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. Let A be a set of indices (not necessarily countable) and let $S = \{|\psi_\alpha\rangle : \alpha \in A\} \subseteq \mathcal{H}$ be a set of normalized pure states. This is equivalent to say that each $|\psi_\alpha\rangle \in S$ is of the form $|\psi_\alpha\rangle = \sum_{i=1}^{d} \psi_{\alpha_i}|\psi_i\rangle$, where $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_d\rangle\}$ is an orthonormal basis of $\mathcal{H}$ and $\psi_{\alpha_1}, \psi_{\alpha_2}, \ldots, \psi_{\alpha_d}$ are complex numbers such that $\sum_{i=1}^{d} |\psi_{\alpha_i}|^2 = 1$. Suppose that we want to gather information about an unknown state $|\psi_x\rangle \in S$. Once we have chosen, and fixed, an orthonormal basis of $\mathcal{H}$, say $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_d\rangle\}$, information about the coefficients $\psi_{x_1}, \psi_{x_2}, \ldots, \psi_{x_d}$ is obtained by performing measurements on the state $|\psi_x\rangle$. The mathematical description of a general measurement on a quantum state is the *Positive Operator Valued Measurement* (POVM) formalism. This is described as follows. Let Λ be a set of indices (not necessarily finite). A POVM $\mathcal{M} = \{\widehat{E}_r : r \in \Lambda\}$ on $\mathcal{H}$ is a set of positive operators $\widehat{E}_r : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\sum_{r\in\Lambda} \widehat{E}_r = \widehat{I}_{\mathcal{H}}$, where $\widehat{I}_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$. The probability that the rth measurement outcome occurs is given by $\langle\psi_x|\widehat{E}_r|\psi_x\rangle$ and the state immediately after the measurement is $\left(\langle\psi_x|\widehat{E}_r^\dagger\widehat{E}_r|\psi_x\rangle\right)^{-\frac{1}{2}}\widehat{E}_r|\psi_x\rangle$ (where $|\psi_x\rangle$ is the state before the measurement). For all practical purposes Λ is taken to be finite. In such a case, since the Hilbert space $\mathcal{H}$ is finite dimensional, it follows by a theorem of Davies [1] that the elements of $\mathcal{M}$ can be chosen to be of rank 1. In this paper, we consider POVMs with rank-1 elements only. As a matter of fact full information about $\psi_{x_1}, \psi_{x_2}, \ldots, \psi_{x_d}$ is obtained only by performing measurements on an infinite number of copies of $|\psi_x\rangle$. Since it is physically impossible to be supplied with an infinite number of copies of a quantum state, we assume that we are supplied with n copies of $|\psi_x\rangle$ only. Now, let A' be a set of indices (not necessarily countable) and let $S_n = \{|\Psi_{\alpha'}\rangle : \alpha' \in A'\} \subseteq \mathcal{H}^{\otimes n}$ be a set of normalized pure states in $\mathcal{H}^{\otimes n}$. We assume that there is a bijective function $f : A \longrightarrow A'$. A *state estimation strategy* $(\mathcal{M}, T)$ is composed of the following:

- A POVM on $\mathcal{H}^{\otimes n}$, $\mathcal{M} = \{\widehat{E}_r : r \in \Lambda\}$;
- A set of density matrices $T = \{\rho_r : r \in \Lambda\} \subseteq S$.

If the rth outcome of a measurement performed by applying $\mathcal{M}$ to a given $|\Psi_{\alpha'}\rangle \in S_n$ occurs, the system is then prepared in the state ρ_r. This is said to be the *estimated state* from the rth measurement outcome. For any $|\Psi_{\alpha'}\rangle \in S_n$, the *average estimated state* of the system is given by

$$\rho^{(\Psi_{\alpha'})} := \sum_{r\in\Lambda} \langle\Psi_{\alpha'}|\widehat{E}_r|\Psi_{\alpha'}\rangle\rho_r.$$

Thus, the *fidelity* for $(\mathcal{M}, T)$ to estimate the state $|\psi_{f^{-1}(\alpha')}\rangle$ is $\langle\psi_{f^{-1}(\alpha')}|\rho^{(\Psi_{\alpha'})}|\psi_{f^{-1}(\alpha')}\rangle$, and the *average fidelity* for $(\mathcal{M}, T)$ to estimate states in S is given by

$$\overline{F}(\mathcal{M}, T) := \int_{\alpha'\in A'} \langle\psi_{f^{-1}(\alpha')}|\rho^{(\Psi_{\alpha'})}|\psi_{f^{-1}(\alpha')}\rangle \, \mathrm{d}(\alpha'), \tag{1}$$

where $d(\alpha')$ is a generalized measure over A'. Then, using the above expression for $\rho^{(\Psi_{\alpha'})}$, we have

$$\overline{F}(\mathcal{M}, T) = \int_{\alpha'\in A'} \sum_{r\in\Lambda} \langle\Psi_{\alpha'}|\widehat{E}_r|\Psi_{\alpha'}\rangle\langle\psi_{f^{-1}(\alpha')}|\rho_r|\psi_{f^{-1}(\alpha')}\rangle \, \mathrm{d}(\alpha'). \tag{2}$$

In this last equation, we can replace $\langle\psi_{f^{-1}(\alpha')}|\rho_r|\psi_{f^{-1}(\alpha')}\rangle$ with a [0, 1]-valued parameter depending on $\mathcal{M}$, r and α'. Such a parameter, denoted by $s(\mathcal{M}, r, \alpha')$, is called the *score*. We write $\overline{F}(\mathcal{M}, s)$ if we want to stress that the average fidelity depends also on a specified score $s(\mathcal{M}, r, \alpha')$. Note that T depends on the choice of s. Let

$$\overline{F}_s^{\max} := \sup_{\mathcal{M}} \overline{F}(\mathcal{M}, s).$$

Our task is to evaluate $\overline{F}_s^{\max}$ and to determine which state estimation strategies achieve this quantity. We simply write $\overline{F}^{\max}$ if the score is taken to be $\langle\psi_{f^{-1}(\alpha')}|\rho_r|\psi_{f^{-1}(\alpha')}\rangle$.

2.2. *How to evaluate* $\overline{F}_s^{\max}$

Let us denote by $\mathcal{L}(A)$ the linear span of a set of vectors A. Let $\{|\Psi_i\rangle : i = 1, \ldots, N\}$ be an orthonormal basis of $\mathcal{L}(S_n)$. We can attain higher values of $\overline{F}(\mathcal{M}, s)$ if we restrict each POVM element $\widehat{E}_r$ to have support in $\mathcal{L}(S_n)$ instead of $\mathcal{H}^{\otimes n}$ (recall that the *support* of an operator is the linear span of its range). In fact, since $|\Psi_{\alpha'}\rangle \in \mathcal{L}(S_n)$, one can get higher value of $\langle\Psi_{\alpha'}|\widehat{E}_r|\Psi_{\alpha'}\rangle$ if every $\widehat{E}_r$ has support in $\mathcal{L}(S_n)$. However, note that if the elements of a POVM have support in a subspace of $\mathcal{H}^{\otimes n}$ containing $\mathcal{L}(S_n)$, then the POVM may still give rise to $\overline{F}_s^{\max}$ (an example is given in section 3.3 below). In order to compute $\overline{F}(\mathcal{M}, s)$, we take the POVM $\mathcal{M} = \{\widehat{E}_r : r \in \Lambda\}$ such that, for every $r \in \Lambda$, we have $\widehat{E}_r = C_r P\big[\sum_{i=1}^N \lambda_{ir}|\Psi_i\rangle\big]$, with the following constraints:

(A) $C_r > 0$ for every $r \in \Lambda$;
(B) $\sum_{i=1}^N |\lambda_{ir}|^2 = 1$, for every $r \in \Lambda$.

Then each $\widehat{E}_r$ has support in $\mathcal{L}(S_n)$. These operators form a POVM on $\mathcal{L}(S_n)$ if and only if

$$\sum_{r\in\Lambda} C_r P\left[\sum_{i=1}^N \lambda_{ir}|\Psi_i\rangle\right] = I_{\mathcal{L}(S_n)} = \sum_{i=1}^N P[|\Psi_i\rangle], \qquad \text{that is, if and only if,}$$

(C) $\sum_{r\in\Lambda} C_r \lambda_{ir}\lambda_{jr}^* = \delta_{ij}$ for every $i, j = 1, \ldots, N$.

Now, the given state of S_n can be written as $|\Psi_{\alpha'}\rangle = \sum_{i=1}^N \mu_i(\alpha')|\Psi_i\rangle$, where

(D) $\sum_{i=1}^N |\mu_i(\alpha')|^2 = 1$.

Note that, although the set $\{(\mu_1(\alpha'), \mu_2(\alpha'), \ldots, \mu_N(\alpha')) : \alpha' \in A'\}$ is known, the individual N-tuple $(\mu_1(\alpha'), \mu_2(\alpha'), \ldots, \mu_N(\alpha'))$ is not, since the supplied state $|\Psi_{\alpha'}\rangle$ is unknown. Thus

$$\langle\Psi_{\alpha'}|\widehat{E}_r|\Psi_{\alpha'}\rangle = C_r \left|\sum_{j=1}^N \lambda_{jr}^*\mu_i(\alpha')\right|^2, \qquad \text{for every} \quad r \in \Lambda.$$

It follows that, with the score $\langle\psi_{f^{-1}(\alpha')}|\rho_r|\psi_{f^{-1}(\alpha')}\rangle$, the average fidelity is

$$\overline{F}(\mathcal{M}, T) = \int_{\alpha'\in A'} \left(\sum_{r\in\Lambda} C_r \left|\sum_{j=1}^N \lambda_{jr}^*\mu_i(\alpha')\right|^2 \langle\psi_{f^{-1}(\alpha')}|\rho_r|\psi_{f^{-1}(\alpha')}\rangle\right) \mathrm{d}(\alpha')$$

or, equivalently,

$$\overline{F}(\mathcal{M}, T) = \sum_{r\in\Lambda}\sum_{j,k=1}^N C_r\lambda_{jr}^*\lambda_{kr}\left(\int_{\alpha'\in A'} \mu_j(\alpha')(\mu_k(\alpha'))^*\langle\psi_{f^{-1}(\alpha')}|\rho_r|\psi_{f^{-1}(\alpha')}\rangle\,\mathrm{d}(\alpha')\right). \tag{3}$$

Our task is to maximize $\overline{F}(\mathcal{M}, T)$ under the constraints (A), (B), (C) and (D). A general approach makes use of Lagrange multipliers. Unless otherwise stated, we take the estimated state ρ_r to be a pure state $|\varphi_r\rangle\langle\varphi_r|$. Let $\{|\chi_j\rangle : j = 1, \ldots, M\}$ be an orthonormal basis of $\mathcal{L}(S)$. With respect to this basis, we can express the estimated state $|\varphi_r\rangle$ as $|\varphi_r\rangle = \sum_{i=j}^{M} \chi_{jr}|\chi_j\rangle$, with $\sum_{j=1}^{M} |\chi_{jr}|^2 = 1$ for every $r \in \Lambda$. The variables considered are $C_r, \lambda_{ir}, \chi_{jr}$, where $r \in \Lambda, i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$. In this paper, instead of make use of Lagrange multipliers, we adopt an algebraic approach.

3. Estimation of Bloch vectors: previous works

We consider here the simplest case of state estimation, that is the problem of estimating the direction of a pure qubit. In this section we sketch some of the related previous works.

3.1. The Bloch sphere representation

Any state of a quantum system described by a two-dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ is called *qubit*. Spin states of an electron and polarization states of a photon are examples of qubits. Any *pure qubit* is a vector of $\mathcal{H}$. There is an one-to-one correspondence between normalized pure qubits and unit vectors of the Euclidean space $\mathbb{R}^3$. This correspondence (which also valid for normalized mixed qubits) is called the *Bloch sphere representation* of qubits. In this representation, any pure qubit $|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}|1\rangle$, corresponds to a *Bloch vector* $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$. More generally, $|\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)| = \frac{1}{2}(I + \widehat{\mathbf{n}}\cdot\widehat{\sigma})$, where I is the 2×2 identity matrix and $\widehat{\sigma}$ is the vector with x-, y- and z-component as the Pauli spin matrices σ_x, σ_y and σ_z, respectively. We write $|\psi(\theta,\phi)\rangle = |\widehat{\mathbf{n}}\rangle$. Here, $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z corresponding to the eigenvalues 1 and -1, respectively. The state $|\psi(\pi-\theta, \pi+\phi)\rangle = \sin\frac{\theta}{2}|0\rangle - \mathrm{e}^{\mathrm{i}\phi}\cos\frac{\theta}{2}|1\rangle$, corresponding to the Bloch vector $-\widehat{\mathbf{n}}$, is orthogonal to $|\psi(\theta,\phi)\rangle$.

3.2. Peres–Wootters

Let $S = \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$, where $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ and $|\psi_3\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$. Let $S_2 = \{|\psi_1\rangle^{\otimes 2}, |\psi_2\rangle^{\otimes 2}, |\psi_3\rangle^{\otimes 2}\}$ and $s(\mathcal{M}, r, j) = |\langle\varphi_r|\psi_j\rangle|^2$. The state $|\varphi_r\rangle$ is the estimated qubit corresponding to the rth measurement outcome of the general POVM $\mathcal{M} = \{\widehat{E}_r = C_r \cdot P[\lambda_{1r}|00\rangle + \lambda_{2r}|01\rangle + \lambda_{3r}|10\rangle + \lambda_{4r}|11\rangle] : r \in \Lambda\}$ satisfying the constraints (A), (B), (C) and (D). Peres and Wootters [17] gave numerical evidence that measurements with entangled bases $\lambda_{1r}|00\rangle + \lambda_{2r}|01\rangle + \lambda_{3r}|10\rangle + \lambda_{4r}|11\rangle$ can give rise to higher average fidelity compared to the case when the measurement bases are not entangled.

3.3. Massar–Popescu

Massar and Popescu [14] considered $S = \{|\psi(\theta,\phi)\rangle : \theta \in [0,\pi], \phi \in [0, 2\pi)\}$, $S_n = \{|\psi(\theta,\phi)\rangle^{\otimes n} : \theta \in [0,\pi], \phi \in [0,2\pi)\}$ and $s(\mathcal{M}, r, (\theta,\phi)) = |\langle\varphi_r|\psi(\theta,\phi)\rangle|^2 = \frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}_r}{2}$, where $|\varphi_r\rangle = \cos\frac{\theta_r}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\phi_r}\sin\frac{\theta_r}{2}|1\rangle = |\widehat{\mathbf{n}}_r\rangle$. The state $|\varphi_r\rangle$ is the estimated qubit corresponding to the rth measurement outcome of the POVM $\mathcal{M} = \{\widehat{E}_r : r \in \Lambda\}$:

- If $n = 1$ then $\Lambda = \{1, 2\}$, $\widehat{E}_1 = |0\rangle\langle 0|$, $\widehat{E}_1 = |1\rangle\langle 1|$, $|\varphi_1\rangle = |0\rangle$ and $|\varphi_2\rangle = |1\rangle$. For $n = 1$, $\overline{F}^{\max} = \frac{2}{3}$.
- If $n = 2$ then $\Lambda = \{1, 2, 3, 4\}$ and $\widehat{E}_j = P\left[\frac{1}{2}|\psi^-\rangle + \frac{\sqrt{3}}{2}|\widehat{\mathbf{n}}_j\rangle^{\otimes 2}\right]$ for $j = 1, \ldots, 4$, where $\widehat{\mathbf{n}}_1 = (0,0,1)$, $\widehat{\mathbf{n}}_2 = \left(\frac{\sqrt{8}}{3}, 0, -\frac{1}{3}\right)$, $\widehat{\mathbf{n}}_3 = \left(\frac{-\sqrt{2}}{3}, \sqrt{\frac{2}{3}}, -\frac{1}{3}\right)$ and $\widehat{\mathbf{n}}_4 = \left(\frac{-\sqrt{2}}{3}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}\right)$; $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $|\varphi_j\rangle = |\widehat{\mathbf{n}}_j\rangle$ for $j = 1, \ldots, 4$. For $n = 2$, $\overline{F}^{\max} = \frac{3}{4}$.

- If $n > 2$ then $\overline{F}^{\max} = \frac{n+1}{n+2}$, which was obtained by making use of an infinite POVM (i.e., for which Λ is an infinite set) known as *covariant measurement* (see, e.g. [11]).

3.4. Derka–Buzek–Ekert

For the general case considered by Massar and Popescu, Derka *et al* [7] have given a finite POVM such that $\overline{F}^{\max} = \frac{n+1}{n+2}$ for any n. In addition, they also considered $S = \left\{\left|\psi\left(\frac{\pi}{2},\phi\right)\right\rangle : \phi \in [0, 2\pi)\right\}$, with $S_n = \left\{\left|\psi\left(\frac{\pi}{2},\phi\right)\right\rangle^{\otimes n} : \phi \in [0, 2\pi)\right\}$, $\Lambda = \{0, \ldots, n\}$ and $\widehat{E}_r = P\left[\frac{1}{\sqrt{n+1}}\sum_{j=0}^{n} \mathrm{e}^{\frac{2\pi i j}{n+1}}\left|S_j^{(n)}\right\rangle\right]$, where

$$\left|S_j^{(n)}\right\rangle = \frac{1}{\sqrt{\binom{n}{j}}} \sum_{\substack{x_i=0,1\\ 1\leqslant i\leqslant n\\ |\{x_i : x_i=0\}|=j}} |x_1 x_2 \cdots x_n\rangle. \tag{4}$$

The state $\left|S_j^{(n)}\right\rangle$ is the symmetrized n-qubit superposition of j 0's and $(n-j)$ 1's, $|\varphi_r\rangle = \left|\psi\left(\frac{\pi}{2}, \frac{2\pi r}{n+1}\right)\right\rangle$ and

$$\overline{F}^{\max} = \frac{1}{2} + \frac{1}{2^{n+1}} \sum_{i=0}^{n-1} \sqrt{\binom{n}{i}\binom{n}{i+1}}.$$

3.5. Latorre–Pascual–Tarrach

Latorre *et al* [12] considered the case of estimation of qubits for which $S = \{|\psi(\theta,\phi)\rangle : \theta \in [0,\pi], \phi \in [0, 2\pi)\}$, $S_n = \{|\psi(\theta,\phi)\rangle^{\otimes n} : \theta \in [0,\pi], \phi \in [0, 2\pi)\}$ and $s(\mathcal{M}, r, (\theta,\phi)) = \frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}_r}{2}$. The value $\overline{F}^{\max} = \frac{n+1}{n+2}$ corresponds to the estimation strategy which uses a POVM with elements $\widehat{E}_r = C_r P[|\psi(\theta_r,\phi_r)\rangle^{\otimes n}]$, and $|\varphi_r\rangle = |\psi(\theta_r,\phi_r)\rangle$. The table below contains the parameters of the strategy for $2 \leqslant n \leqslant 5$. For $n > 5$, the minimal finite POVM could be established.

n	r	C_r	ϕ_r/π	$\cos\theta_r$	$\overline{F}^{\max}$
2	1 $2-4$	3/4	0 $2(r-2)/3$	1 $-1/3$	3/4
3	1 2 $3-6$	2/3	0 0 $(r-3)/2$	1 -1 0	4/5
4	1 2 $3-6$ $7-10$	5/12 5/12 25/48 25/48	0 0 $(r-3)/2$ $\left(r-\frac{13}{2}\right)/2$	1 -1 $1/\sqrt{5}$ $-1/\sqrt{5}$	5/6
5	1 2 $3-7$ $8-12$	1/2	0 0 $2(r-3)/5$ $2\left(r-\frac{15}{2}\right)/5$	1 -1 $1/\sqrt{5}$ $-1/\sqrt{5}$	5/7

3.6. Gisin–Popescu–Massar

Gisin and Popescu [10] considered the problem of estimating the direction a qubit $|\psi(\theta,\phi)\rangle$ from the entire Bloch sphere, when an *anti-parallel* state $|\psi(\theta,\phi)\rangle \otimes |\psi(\pi-\theta, \pi+\phi)\rangle$

is supplied with equal probability over the set (θ, ϕ). Then, let $S = \{|\psi(\theta, \phi)\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi)\}$ and $S_2 = \{|\psi(\theta, \phi)\rangle \otimes |\psi(\pi - \theta, \pi + \phi)\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi)\}$. The state $|\varphi_r\rangle = |\widehat{\mathbf{n}}_r\rangle$ is the estimated qubit corresponding to the rth measurement outcome of the POVM $\mathcal{M} = \{\widehat{E}_r : r \in \Lambda\}$: $\widehat{E}_r = P\big[\alpha|\widehat{\mathbf{n}}_r, -\widehat{\mathbf{n}}_r\rangle - \beta \sum_{k=1, k\neq r}^{n} |\widehat{\mathbf{n}}_k, -\widehat{\mathbf{n}}_k\rangle\big]$, for $r \in \{1, 2, 3, 4\} = \Lambda$, where $\alpha = \frac{13}{6\sqrt{6}-2\sqrt{2}}$ and $\beta = \frac{5-2\sqrt{3}}{6\sqrt{6}-2\sqrt{2}}$. The average fidelity for this strategy was shown to be $\overline{F}(\mathcal{M}, T) = \frac{5\sqrt{3}+33}{3(3\sqrt{3}-1)^2}$. Massar [13] established that this strategy is optimal. Moreover, Massar proved that in order to estimate the direction of a vector that lies on the plane perpendicular to the direction of the Bloch vector of the qubit $|\psi(\theta, \phi)\rangle$ (so $s(\mathcal{M}, T, r, (\theta, \phi)) = 1 - (\widehat{\mathbf{n}} \cdot \widehat{\mathbf{n}}_r)^2$), parallel and anti-parallel states give the optimal fidelities $\overline{F}_s^{\max} = 0.8$ and $\overline{F}_s^{\max} = 0.733$, respectively. Thus, parallel qubits give then better fidelity for the chosen score.

3.7. LOCC measurements

Let us suppose that we are restricted to perform measurements on individual qubits and then use the measurements results for one qubit to perform measurements on another qubit and so on and so forth (this procedure is known as *LOCC measurement*). For every encoding, the supplied multiqubit state would then contain the same amount of information about the direction of the qubit as far as encoding is done in terms of product states. One can ask: what kind of measurement provides more information about the direction of the qubit (LOCC or an entangled one)? Gill and Massar [9] have shown that, as $n \to \infty$, the difference between optimal fidelities for LOCC and entangled measurements (on n copies of the qubit) goes to zero. This is true not only for encodings of the form $|\psi\rangle^{\otimes n}$, but also for any other kind of product state encoding. For similar results see also Bagan *et al* [3].

4. Estimation of parallel and anti-parallel qubits

We consider here the problem of estimating the direction of pure qubit taken from a circle (for given θ), when n copies of the qubit are supplied with equal probability. The qubit which we are going to estimate belongs then to the set $S_\theta = \{|\psi(\theta, \phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle : \phi \in [0, 2\pi)\}$. The set of supplied states is $S_n^{(\theta)} = \{|\psi(\theta, \phi)\rangle^{\otimes n} : \phi \in [0, 2\pi)\}$, where $|\varphi_r\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi_r}\sin\frac{\theta}{2}|1\rangle = |\widehat{\mathbf{n}}_r\rangle$. Figure 2 illustrates this setting.

The score is $s(\mathcal{M}, r, (\theta, \phi)) = |\langle\varphi_r|\psi(\theta, \phi)\rangle|^2 = \frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}_r}{2}$. The elements of $S_n^{(\theta)}$ can be written as

$$|\psi(\theta, \phi)\rangle^{\otimes n} = \sum_{j=0}^{n} \sqrt{\binom{n}{j}} \left(\cos\frac{\theta}{2}\right)^j \left(\sin\frac{\theta}{2}\right)^{n-j} e^{i(n-j)\phi} \big|S_j^{(n)}\big\rangle = |\Psi_{n,0}(\theta, \phi)\rangle,$$

where $\big|S_j^{(n)}\big\rangle$ is expressed in (4). More generally, for any fixed $\theta \in [0, \pi]$, for estimating the direction of $|\psi(\theta, \phi)\rangle \in S_\theta$, we can consider the scenario in which the state is supplied with equal probability from the set

$$S_{n,m}^{(\theta)} = \{|\psi(\theta, \phi)\rangle^{\otimes n} \otimes |\psi(\pi - \theta, \pi + \phi)\rangle^{\otimes m} : \phi \in [0, 2\pi)\}. \tag{5}$$

We use the notation

$$|\Psi_{n,m}(\theta, \phi)\rangle = |\psi(\theta, \phi)\rangle^{\otimes n} \otimes |\psi(\pi - \theta, \pi + \phi)\rangle^{\otimes m}.$$

Again, we take here $|\varphi_r\rangle = |\psi(\theta, \phi_r)\rangle$. Any state of $S_{n,m}^{(\theta)}$ can be then written as

$$|\Psi_{n,m}(\theta, \phi)\rangle = \sum_{p=0}^{n+m} e^{i(n+m-p)\phi} \mathcal{N}_p(\theta) |\xi_p(\theta)\rangle,$$

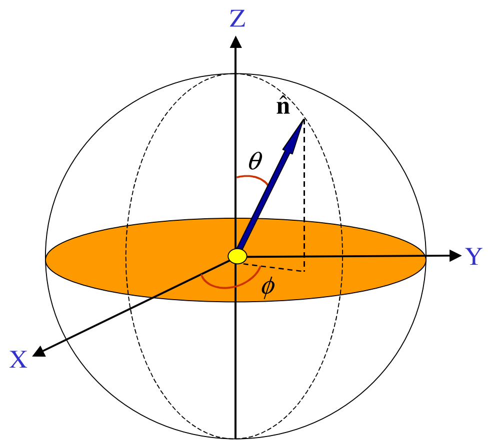

Figure 1.

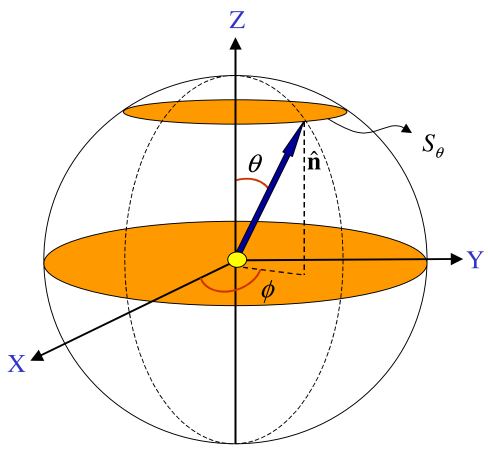

Figure 2.

where

$$|\xi_p(\theta)\rangle = \frac{1}{\mathcal{N}_p(\theta)} \sum_{(k,l)\in T_p}^{n+m} \sqrt{\binom{n}{k}\binom{m}{l}} \left(\cos\frac{\theta}{2}\right)^{2(m-l+k)} \left(\sin\frac{\theta}{2}\right)^{2(n+l-k)} (-1)^{m-l} \left|S_k^{(n)}\right\rangle \otimes \left|S_l^{(m)}\right\rangle,$$

$$\mathcal{N}_p(\theta) = \left(\sum_{(k,l)\in T_p}^{n+m} \binom{n}{k}\binom{m}{l} \left(\cos\frac{\theta}{2}\right)^{2(m-l+k)} \left(\sin\frac{\theta}{2}\right)^{2(n+l-k)} \right)^{\frac{1}{2}}$$

and

$$T_p = \{(k,l) \in \{0,\ldots,n\} \times \{0,\ldots,m\} : k+l = p\}.$$

Following the description given in section 3, the elements of the most general POVM, which may appear in an estimation strategy are of the form

$$\widehat{E}_r = C_r^{(\theta)} P \left[\sum_{p=0}^{n+m} \lambda_{rp}(\theta) |\xi_p(\theta)\rangle \right], \qquad \text{for every} \quad r \in \Lambda,$$

where $C_r^{(\theta)} > 0$, $\sum_{p=0}^{n+m} |\lambda_{rp}(\theta)|^2 = 1$ for every $r \in \Lambda$, and

$$\sum_{r\in\Lambda} C_r^{(\theta)} \lambda_{rp}(\theta)(\lambda_{rq}(\theta))^* = \delta_{pq}, \qquad \text{for all} \quad p, q \in \{0, 1, \ldots, n+m\}. \tag{6}$$

The average fidelity corresponding to this estimation strategy will be denoted by $\overline{F}_{n,m}(\theta)$. Using the POVM described in the previous section, we obtain

$$\overline{F}_{n,m}(\theta) = \frac{1+\cos^2\theta}{2} + \frac{\sin^2\theta}{2} \sum_{p=0}^{n+m} \mathcal{N}_{p-1}(\theta)\mathcal{N}_p(\theta) \sum_{r\in\Lambda} C_r^{(\theta)} \operatorname{Re}(\lambda_{r(p-1)}(\theta)(\lambda_{rp}(\theta))^* \,\mathrm{e}^{-\mathrm{i}\phi_r})$$

and we observe that

$$\overline{F}_{n,m}(\theta) \leqslant \frac{1+\cos^2\theta}{2} + \frac{\sin^2\theta}{2} \sum_{p=0}^{n+m} \mathcal{N}_{p-1}(\theta)\mathcal{N}_p(\theta) \sum_{r\in\Lambda} C_r^{(\theta)} |\lambda_{r(p-1)}(\theta) \times \lambda_{rp}(\theta)|. \tag{7}$$

Then by the Schwartz inequality,

$$\begin{aligned}\overline{F}_{n,m}(\theta) &\leqslant \frac{1+\cos^2\theta}{2} + \frac{\sin^2\theta}{2} \sum_{p=1}^{n+m} \mathcal{N}_{p-1}(\theta)\mathcal{N}_p(\theta) \\ &\quad\times \left[\sum_{r\in\Lambda} C_r^{(\theta)} |\lambda_{r(p-1)}(\theta)|^2\right]^{\frac{1}{2}} \left[\sum_{r\in\Lambda} C_r^{(\theta)} |\lambda_{rp}(\theta)|^2\right]^{\frac{1}{2}} \\ &= \frac{1+\cos^2\theta}{2} + \frac{\sin^2\theta}{2} \sum_{p=1}^{n+m} \mathcal{N}_{p-1}(\theta)\mathcal{N}_p(\theta),\end{aligned} \tag{8}$$

which follows from (6). We then see that

$$\overline{F}_{n,m}^{\max}(\theta) \leqslant \frac{1+\cos^2\theta}{2} + \frac{\sin^2\theta}{2} \sum_{p=1}^{n+m} \mathcal{N}_{p-1}(\theta)\mathcal{N}_p(\theta),$$

which is an upper bound on $\overline{F}_{n,m}^{\max}(\theta)$ independent of any measurement strategy. We describe now an estimation strategy which attains this quantity. Equality in (8) holds if and only if

$$C_r^{(\theta)} |\lambda_{r(p-1)}(\theta)|^2 = K_p C_r^{(\theta)} |\lambda_{rp}(\theta)|^2, \qquad \text{for every} \quad r \in \Lambda,$$

where K_p is constant for $p = 0, 1, \ldots, n+m$. It follows from condition (6) that $K_p = 1$ for every $p = 0, 1, \ldots, n+m$. This implies that

$$\lambda_{rp}(\theta) = \frac{\mathrm{e}^{\mathrm{i}\varepsilon_{rp}}}{\sqrt{n+m+1}}, \qquad \text{where} \quad \varepsilon_{rp} \in \mathbb{R} \text{ for every } p = 0, 1, \ldots, n+m \text{ and } r \in \Lambda. \tag{9}$$

Using (9), we see that equality in (7) holds if and only if $\varepsilon_{rp} = 2n_{rp}\pi + \varepsilon_{r(p+1)} + \phi_r$, for each $r \in \Lambda$ and each $p = 0, 1, \ldots, n+m$, where $n_{rp} \in \mathbb{Z}$. Then

$$\varepsilon_{rp} = 2L_{rp}\pi + \varepsilon_{r(n+m)} + (n+m-p)\phi_r, \qquad \text{where} \quad L_{rp} \in \mathbb{Z} \text{ for every } r \in \Lambda \text{ and } p = 0, 1, \ldots, n+m. \tag{10}$$

Using (9) and (10) into (6), we can write

$$\sum_{r\in\Lambda} C_r^{(\theta)} \,\mathrm{e}^{\mathrm{i}(q-p)\phi_r} = (m+n+1)\delta_{pq}, \qquad \text{for every} \quad p, q = 0, 1, \ldots, n+m. \tag{11}$$

Thus, we see that one possible situation where condition (11) is satisfied is given by $\Lambda = \{0, 1, \ldots, n+m\}$, $C_r^{(\theta)} = 1$ for all $r \in \Lambda$, and $\phi_r = \frac{2\pi r}{n+m+1}$ for all $r \in \Lambda$. Taking these parameters in the estimation strategy, we define a POVM $\mathcal{M} = \{\widehat{E}_r : r \in \Lambda\}$ such that

$$\widehat{E}_r = C_r^{(\theta)} P\left[\sum_{p=0}^{n+m} \lambda_{rp}(\theta)|\xi_p(\theta)\rangle\right] = P\left[\frac{1}{\sqrt{m+m+1}} \sum_{p=0}^{n+m} \exp\left[\frac{2\pi\mathrm{i}(n-m-p)r}{n+m+1}\right] |\xi_p(\theta)\rangle\right].$$

Then

$$\overline{F}_{n,m}(\theta) = \frac{1+\cos^2\theta}{2} + \frac{\sin^2\theta}{2}\sum_{p=1}^{n+m}\mathcal{N}_{p-1}(\theta)\mathcal{N}_p(\theta) = \overline{F}^{\max}_{n,m}(\theta).$$

Note that the basis of the POVM is the Fourier basis of dimension $n + m + 1$.

4.1. The dimensional argument

Gisin and Popescu [10] have shown that the anti-parallel qubits $|\Psi_{1,1}(\theta,\phi)\rangle$ contain more information on an average compared to parallel qubits $|\Psi_{2,0}(\theta,\phi)\rangle$, regarding the direction of the qubit $|\psi(\theta,\phi)\rangle$, when (θ,ϕ) is uniformly distributed over $[0,\pi]\times[0,2\pi)$. This is counterintuitive according to the reasoning in classical physics. In fact, in order to get information about the direction of a classical vector $\mathbf{v}$, either we can consider the parallel vectors $\{\mathbf{v},\mathbf{v}\}$ or the anti-parallel vectors $\{\mathbf{v},-\mathbf{v}\}$ (when we are restricted to only to these two types of vectors). The parallel and the anti-parallel vectors do not make any difference in this regard. This is simply because $\mathbf{v}$ and $-\mathbf{v}$ contains the same information about the direction of $\mathbf{v}$. However, a notable property of parallel and anti-parallel qubits is the following: $\dim\mathcal{L}(\{|\Psi_{1,1}(\theta,\phi)\rangle : \theta\in[0,\pi],\phi\in[0,2\pi)\}) = 4$ and $\dim\mathcal{L}(\{|\Psi_{2,0}(\theta,\phi)\rangle : \theta\in[0,\pi],\phi\in[0,2\pi)\}) = 3$. Gisin and Popescu proposed the difference in dimension as the reason behind the difference in optimal fidelities. We call this reasoning as *dimensional argument*. They support the dimensional argument as follows. Even though

$$|\langle\Psi_{1,1}(\theta,\phi)|\Psi_{1,1}(\theta',\phi')\rangle| = |\langle\Psi_{2,0}(\theta,\phi)|\Psi_{2,0}(\theta',\phi')\rangle|, \quad \text{where} \quad \theta,\theta'\in[0,\pi] \text{ and } \phi,\phi'\in[0,2\pi),$$

anti-parallel states are, as a *whole*, *farther apart* than parallel states, because of the difference on the dimensions of the linear spans. Note that

$$\begin{aligned}|\langle\Psi_{1,1}(\theta,\phi)|\Psi_{1,1}(\theta',\phi')\rangle|^2 &= |\langle\psi(\theta,\phi)|\psi(\theta',\phi')\rangle|^2\cdot|\langle\psi(\pi-\theta,\pi+\phi)|\psi(\pi-\theta',\pi+\phi')\rangle|^2\\ &= \left(\frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}'}{2}\right)\cdot\left(\frac{1+(-\widehat{\mathbf{n}})\cdot(-\widehat{\mathbf{n}}')}{2}\right) = \left(\frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}'}{2}\right)^2\\ &= |\langle\Psi_{2,0}(\theta,\phi)|\Psi_{2,0}(\theta',\phi')\rangle|^2,\end{aligned}$$

where $|\psi(\theta,\phi)\rangle = |\widehat{\mathbf{n}}\rangle$ and $|\psi(\theta',\phi')\rangle = |\widehat{\mathbf{n}}'\rangle$. Figure 3 clarifies the meaning of *farther apart*. The three unit vectors $\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2$ and $\widehat{\mathbf{n}}_3$ lie on the equatorial plane and are linearly dependent. The angle between each pair of them is $\alpha = \frac{2}{3}\pi$. We consider now three linearly independent vectors $\widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_2$ and $\widehat{\mathbf{m}}_3$, whose heads are on a small circle such that the great circle joining north pole and the head of $\widehat{\mathbf{n}}_i$ crosses the equator in the head of $\widehat{\mathbf{m}}_i$. This means that the angle between $\widehat{\mathbf{m}}_i$ and $\widehat{\mathbf{m}}_j$ is smaller than the angle α between $\widehat{\mathbf{n}}_i$ and $\widehat{\mathbf{n}}_j$. In order to make the angle between $\widehat{\mathbf{m}}_i$ and $\widehat{\mathbf{m}}_j$ to be equal to α we need to rotate them in such a way that the distance between their heads increases.

The mathematical formulation distilled from the above argument can be described as follows:

Problem. *Let $\mathcal{H}$ be a d-dimensional Hilbert space. Let Δ be a set of indices (not necessarily finite). Let $\mathbb{E}_{|\Phi\rangle} = \{(|\Phi_i\rangle, p_i : |\Phi_i\rangle\in\mathcal{H}, 0\leqslant p_i\leqslant 1$ for every $i\in\Delta$ and $\sum_{i\in\Delta}p_i = 1\}$ and $\mathbb{E}_{|\Upsilon\rangle} = \{(|\Upsilon_i\rangle, p_i) : |\Upsilon_i\rangle\in\mathcal{H}, 0\leqslant p_i\leqslant 1$ for every $i\in\Delta$ and $\sum_{i\in\Delta}p_i = 1\}$. Suppose that $|\langle\Phi_i|\Phi_j\rangle| = |\langle\Upsilon_i|\Upsilon_j\rangle|$ for every $i,j\in\Delta$, and that $\dim\mathcal{L}(\{|\Phi_i\rangle : i\in\Delta\}) > \dim\mathcal{L}(\{|\Upsilon_i\rangle : i\in\Delta\})$. Then $\overline{F}^{\max}$ for estimating i, for states given from $\mathbb{E}_{|\Phi\rangle}$ should be grater than that for states given from $\mathbb{E}_{|\Upsilon\rangle}$.*

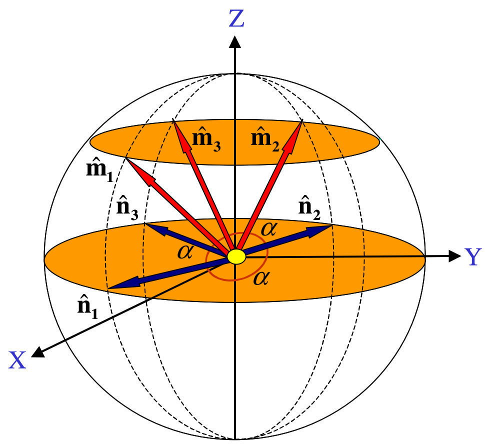

Figure 3.

The problem can be restated in a more concrete form as follows. Let $s : \Delta \times \Lambda \longrightarrow [0, 1]$ be the score when a POVM $\mathcal{M} = \{E_r : r \in \Lambda\}$ is applied on the unknown state $|\Phi_i\rangle$, given from the set $\mathbb{E}_{|\Phi\rangle}$, with probability p_i, and the rth outcome has occurred. Note that the set of values of the score s are different for different forms of Λ (i.e., for different choices of the POVM $\mathcal{M}$). Then the average fidelity is

$$\overline{F}(\mathcal{M}, \mathbb{E}_{|\Phi\rangle}, s) = \sum_{r\in\Lambda}\sum_{i\in\Delta} p_i \langle\Phi_i|E_r|\Phi_i\rangle s(i, r).$$

Similarly, for $\mathbb{E}_{|\Upsilon\rangle}$, we have $\overline{F}(\mathcal{M}, \mathbb{E}_{|\Upsilon\rangle}, s) = \sum_{r\in\Lambda}\sum_{i\in\Delta} p_i\langle\Upsilon_i|E_r|\Upsilon_i\rangle s(i, r)$. Let $\overline{F}^{\max}(\mathbb{E}_{|\Phi\rangle}, s)$ be the maximum of $\overline{F}(\mathcal{M}, \mathbb{E}_{|\Phi\rangle}, s)$ over all possible choices of the POVM $\mathcal{M}$ ($\overline{F}^{\max}(\mathbb{E}_{|\Upsilon\rangle}, s)$ is defined similarly). Suppose that the following conditions are satisfied simultaneously:

- $|\langle\Phi_i|\Phi_j\rangle| = |\langle\Upsilon_i|\Upsilon_j\rangle|$, for every $i, j \in \Delta$;
- $\dim \mathcal{L}(\{|\Phi_i\rangle : i \in \Delta\}) > \dim \mathcal{L}(\{|\Upsilon_i\rangle : i \in \Delta\})$;
- $\mathcal{L}(\{|\Phi_i\rangle : i \in \Delta\}) \supset \mathcal{L}(\{|\Upsilon_i\rangle : i \in \Delta\})$.

Then we need to prove that $\overline{F}^{\max}(\mathbb{E}_{|\Phi\rangle}, s) > \overline{F}^{\max}(\mathbb{E}_{|\Upsilon\rangle}, s)$. A solution to this problem is still missing. So we do not know whether the statement of the above problem can be taken as general principle.

4.1.1. Entropic argument. Consider the average density matrices

$$\overline{\rho}_{2,0} = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} P[|\Psi_{2,0}(\theta, \phi)\rangle] \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

and

$$\overline{\rho}_{1,1} = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} P[|\Psi_{1,1}(\theta, \phi)\rangle] \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi,$$

associated with the ensembles for parallel and anti-parallel states

$$S_{2,0}^{(\theta)} = \{|\Psi_{2,0}(\theta, \phi)\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi)\}$$

and

$$S_{1,1}^{(\theta)} = \{|\Psi_{1,1}(\theta, \phi)\rangle : \theta \in [0, \pi], \phi \in [0, 2\pi)\}.$$

Let $S(\rho)$ the *von Neumann entropy* of a density matrix ρ. This is defined as $S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$, where λ_i is the ith eigenvalue of ρ. The von Neumann entropy is a measure of the information content of a density matrix. One can check that $S(\overline{\rho}_{1,1}) > S(\overline{\rho}_{2,0})$; therefore anti-parallel states can be better distinguished, and hence, they posses more information about the qubit. It should be noted that, even if the qubits $|\psi(\theta,\phi)\rangle$ belong to the circle S_θ, the von Neumann entropy $S(\overline{\rho}_{1,1}(\theta))$ of the average density matrix of the supplied anti-parallel qubits is greater than or equal to the von Neumann entropy $S(\overline{\rho}_{2,0}(\theta))$ of the average density matrix of the supplied parallel qubits (see figure 5, right). This argument does not hold in general. We provide a counterexample. Consider the following two ensembles:

$$\mathcal{E}_1 = \left\{|0\rangle, \frac{1}{2} + \frac{\sqrt{2}+1/2}{4}; |1\rangle, \frac{1}{2} - \frac{\sqrt{2}+1/2}{4}\right\} \quad \text{and} \quad \mathcal{E}_2 = \left\{|0\rangle, \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{2}\right\}.$$

Then

$$S\left(\frac{1}{2}P[|0\rangle] + \frac{1}{2}P\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]\right) = H\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right) > S\left(\left(\frac{1}{2} + \frac{\sqrt{2}+1/2}{4}\right)P[|0\rangle] + \left(\frac{1}{2} - \frac{\sqrt{2}+1/2}{4}\right)P[|1\rangle]\right) = H\left(\frac{1}{2} - \frac{\sqrt{2}+1/2}{4}\right),$$

where

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x) \qquad \text{for} \quad 0 \leqslant x \leqslant 1.$$

This shows that, even though the states in the ensemble $\mathcal{E}_1$ are better distinguished than the states in $\mathcal{E}_2$ (as the states in $\mathcal{E}_1$ are orthogonal to each other and the states in $\mathcal{E}_2$ are not), the information content of the density matrix corresponding to $\mathcal{E}_1$ is less than that the one corresponding to $\mathcal{E}_2$.

4.2. Inadequacy of the dimensional argument

First of all, observe that, for any fixed $\theta \in (0,\pi)$, we have seen that so far as $(n+m)$ is fixed, the $\mathcal{L}(S_{n,m}^{(\theta)})$ is $(n+m+1)$-dimensional subspace of the 2^{n+m}-dimensional Hilbert space of the system. Of course the subspace $\mathcal{L}(S_{n,m}^{(\theta)})$ is different for different values of n and m. Thus, if $\overline{F}_{n,m}^{\max}(\theta) \neq \overline{F}_{n',m'}^{\max}(\theta)$, for $n+m = n'+m'$, the dimensionality argument cannot be used to explain this difference. We provide here three cases: (1) Let $n+m=2$. Then (n,m) is either $(2,0)$, $(1,1)$ or $(0,2)$. (2) Let $n+m=3$. Then (n,m) is either $(3,0)$, $(2,1)$, $(1,2)$ or $(0,3)$. (3) Let $n+m=4$. Then $(4,0)$, $(3,1)$, $(2,2)$, $(1,3)$ or $(0,4)$. Note that $\overline{F}_{n,m}(\theta) \neq \overline{F}_{m,n}(\theta)$, since by swapping we can obtain the state $|\psi(\theta,\phi)\rangle^{\otimes n} \otimes |\psi^\perp(\theta,\phi)\rangle^{\otimes m}$ from $|\psi^\perp(\theta,\phi)\rangle^{\otimes n} \otimes |\psi(\theta,\phi)\rangle^{\otimes m}$. These cases are illustrated by figure 4 ((1) left and (2) right) and figure 5 ((3) left). In none of these three figures the minimum of $\overline{F}_{n,0}^{\max}(\theta)$ is attained at $\theta = \pi/2$. This is attained at two points symmetrically about $\pi/2$. This phenomenon is somehow unexpected. As the circle S_θ is going far and far from the poles towards the equator, we lose more and more information about the direction of $|\psi\rangle$ (in S_θ). It is then expected that the optimal fidelity for states in S_θ (when the supplied state is of the form $|\psi(\theta,\phi)\rangle^{\otimes n} \otimes |\psi(\pi-\theta,\pi+\phi)\rangle^{\otimes m}$) would start to decrease from $\theta = 0$, attaining its minimum at $\theta = \pi/2$, and again start to increase, attaining its maximum at $\theta = \pi$.

5. Estimation of qubits from two circles

Consider the problem of estimating the direction of the Bloch vector $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \sin\theta)$ of a qubit $|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}|1\rangle$, contained within

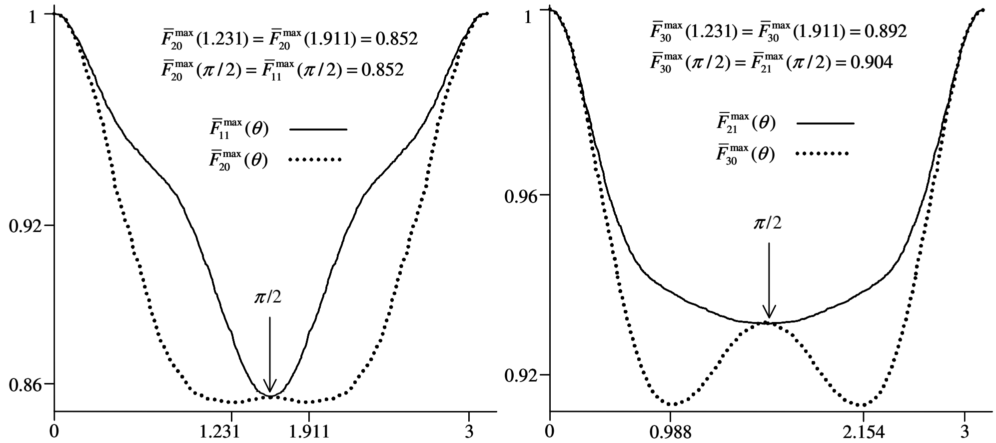

Figure 4.

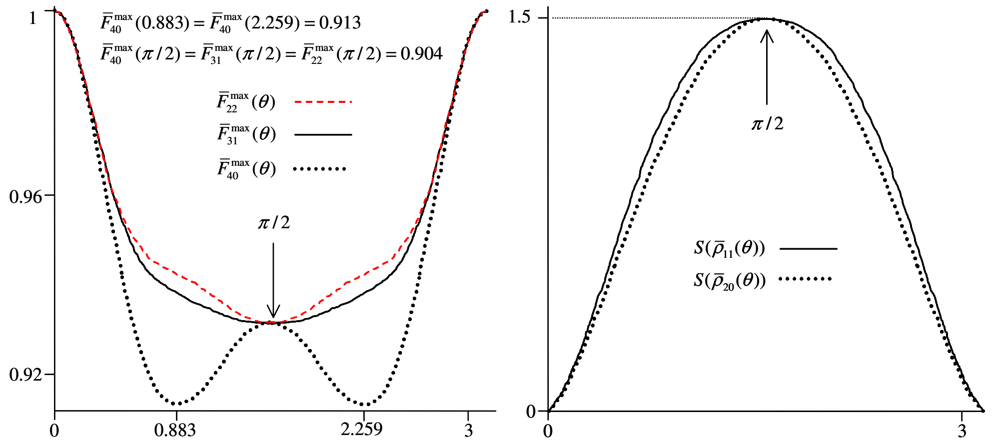

Figure 5.

the circle $S_\theta = \{|\psi(\theta,\phi)\rangle : \phi \in [0, 2\pi)\}$, where $\theta \in [0, \pi]$ is arbitrary but fixed. We have seen that in the case of estimating the direction of the Bloch vector of the qubit $|\psi(\theta,\phi)\rangle \in S_\theta$, the anti-parallel qubits $|\Psi_{1,1}\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\pi-\theta, \pi+\phi)\rangle$ give better information compared to the parallel qubits $|\Psi_{2,0}\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\theta,\phi)\rangle$ (where ϕ is uniformly distributed over $[0.2\pi)$). A generalization of these two kinds of encodings of the initial two qubits is of the following form. The supplied two-qubit state is of the form

$$|\Psi(\theta,\theta_0,\phi)\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\theta+\theta_0,\phi)\rangle, \tag{12}$$

where θ_0 is an arbitrary but fixed element from the set $[-\theta, \pi-\theta]$, and ϕ is uniformly distributed over $[0, 2\pi)$ (see figure 6).

One can check that

$$|\Psi(\theta,\theta_0,\phi)\rangle = \cos\frac{\theta}{2}\cos\frac{\theta+\theta_0}{2}|00\rangle + \mathrm{e}^{2\mathrm{i}\phi}\sin\frac{\theta}{2}\sin\frac{\theta+\theta_0}{2}|11\rangle + \mathrm{e}^{\mathrm{i}\phi}N(\theta,\theta_0)|\chi(\theta,\theta_0)\rangle, \tag{13}$$

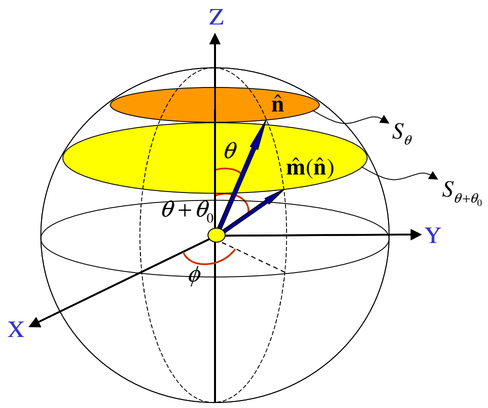

Figure 6.

where

$$|\chi(\theta,\theta_0)\rangle = \frac{1}{N(\theta,\theta_0)}\left(\cos\frac{\theta}{2}\sin\frac{\theta+\theta_0}{2}|01\rangle + \sin\frac{\theta}{2}\cos\frac{\theta+\theta_0}{2}|10\rangle\right),$$
$$N(\theta,\theta_0) = \sqrt{\cos^2\frac{\theta}{2}\sin^2\frac{\theta+\theta_0}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{\theta+\theta_0}{2}}. \tag{14}$$

Thus, varying ϕ over $[0, 2\pi)$, we see that for fixed θ and θ_0, the set (which we denote as $S_{1,1}(\theta,\theta_0)$) of all states $|\Psi(\theta,\theta_0,\phi)\rangle$, given in (13), spans a three-dimensional subspace (with an orthonormal basis $\{|00\rangle, |11\rangle, |\chi(\theta,\theta_0)\rangle\}$) of the total four-dimensional two-qubit Hilbert space.

For estimation, let us choose a POVM $\mathcal{M} = \{E_r : r \in \Lambda\}$ on the linear span $\mathcal{L}\left(S_{1,1}(\theta,\theta_0)\right)$ of $S_{1,1}(\theta,\theta_0)$, where

$$E_r = C_r P\left[\alpha_{1r}|00\rangle + \alpha_{2r}|11\rangle + \alpha_{3r}|\chi(\theta,\theta_0)\rangle\right], \tag{15}$$

with

$$C_r > 0,$$
$$\sum_{j=1}^{3}|\alpha_{jr}|^2 = 1 \qquad \text{for all} \quad r \in \Lambda,$$
$$\sum_{r\in\Lambda} C_r\alpha_{jr}\alpha_{kr}^* = \delta_{jk} \qquad \text{for all} \quad j,k = 1,2,3. \tag{16}$$

The score is

$$s(\mathcal{M}, r, (\theta,\phi)) = |\langle\psi(\theta,\phi)|\psi(\theta,\phi_r)\rangle|^2.$$

So, the average state estimation fidelity is given by

$$\overline{F}_{1,1}(\theta,\theta_0) = \frac{1}{2\pi}\sum_{r\in\Lambda}\int_{\phi=0}^{2\pi}\langle\Psi(\theta,\theta_0,\phi)|E_r|\Psi(\theta,\theta_0,\phi)\rangle|\langle\psi(\theta,\phi)|\psi(\theta,\phi_r)\rangle|^2\,\mathrm{d}\phi$$
$$= 1 - \frac{\sin^2\theta}{2} + \frac{\sin^2\theta}{2}N(\theta,\theta_0)\left\{\cos\frac{\theta}{2}\cos\frac{\theta+\theta_0}{2}\sum_{r\in\Lambda}C_r\mathrm{Re}\{\alpha_{1r}\alpha_{3r}^*\,\mathrm{e}^{\mathrm{i}\phi_r}\}\right.$$
$$\left. + \sin\frac{\theta}{2}\sin\frac{\theta+\theta_0}{2}\sum_{r\in\Lambda}C_r\,\mathrm{Re}\{\alpha_{2r}\alpha_{3r}^*\,\mathrm{e}^{-\mathrm{i}\phi_r}\}\right\}. \tag{17}$$

Thus we see that

$$\overline{F}_{1,1}(\theta,\theta_0) \leqslant 1 - \frac{\sin^2\theta}{2} + \frac{\sin^2\theta\cos\frac{\theta_0}{2}}{2} N(\theta,\theta_0), \tag{18}$$

a quantity independent of the choice of the POVM. Note that as here $-\frac{\pi}{2} \leqslant -\frac{\theta}{2} \leqslant \frac{\theta_0}{2} \leqslant \frac{\pi-\theta}{2} \leqslant \frac{\pi}{2}$, therefore $\cos\frac{\theta_0}{2} \geqslant 0$.

The following is a choice for which equality holds good in (18):

$$\Lambda = \{1,2,3\}, \quad C_r = 1 \quad \text{for all} \quad r \in \Lambda,$$
$$\alpha_{jr} = \frac{1}{\sqrt{3}}\exp\left[\frac{2\pi \mathrm{i}(j-1)(r-1)}{3}\right] \quad \text{for all} \quad r \in \Lambda \text{ and for all} \quad j = 1,2,3, \tag{19}$$

while

$$\phi_1 = 0, \qquad \phi_2 = \frac{4\pi}{3}, \qquad \phi_3 = \frac{2\pi}{3}. \tag{20}$$

Thus we see that the optimal average fidelity, in this case, is given by

$$\overline{F}^{\max}_{1,1}(\theta,\theta_0) = 1 - \frac{\sin^2\theta}{2} + \frac{\sin^2\theta\cos\frac{\theta_0}{2}}{2} N(\theta,\theta_0), \tag{21}$$

where $N(\theta,\theta_0)$ is given in (14). We would like to know now for which value(s) of $\theta_0 \in [-\theta, \pi-\theta]$, $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ is maximum, given any arbitrary but fixed $\theta \in [0,\pi]$. Note that (according to our notations, used in earlier sections)

$$\overline{F}^{\max}_{2,0}(\theta) = \overline{F}^{\max}_{1,1}(\theta,0) \qquad \text{and} \qquad \overline{F}^{\max}_{1,1}(\theta) = \overline{F}^{\max}_{1,1}(\theta,\pi-2\theta).$$

Also note that the maximum value of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ is equal to 1 for both $\theta = 0$ as well as $\theta = \pi$ (irrespective of values of θ_0). Basically, when $\theta = 0$, the state estimation (which we considered here) reduces to estimating the direction of the qubit $|0\rangle$, given the supply of the two-qubit states $|0\rangle \otimes |\psi(\theta_0,\phi)\rangle$ for the uniform distribution of ϕ over $[0,2\pi)$. Hence the optimal fidelity must be 1. Same is the case when $\theta = \pi$. On the other hand, for given any $\theta \in [0,\pi]$, $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ will reach its minimum when $\theta_0 = -\theta$ and $\theta_0 = \pi - \theta$. This is because when $\theta_0 = -\theta$ (or $\theta_0 = \pi-\theta$), the set of supplied two-qubit states is of the form $\{|\psi(\theta,\phi)\rangle \otimes |0\rangle : \phi \in [0,2\pi)\}$ (or $\{|\psi(\theta,\phi)\rangle \otimes |1\rangle : \phi \in [0,2\pi)\}$). And this gives rise to the same optimal average fidelity $\overline{F}^{\max}_{1,1}(\theta,-\theta)$ (= $\overline{F}^{\max}_{1,1}(\theta,\pi-\theta)$) as in the case of estimating the direction of the qubits $|\psi(\theta,\phi)\rangle$ (for fixed θ), when the supplied set of states is the circle $S_\theta = \{|\psi(\theta,\phi)\rangle : \phi \in [0,2\pi)\}$ itself. So, in our notation, we have

$$\begin{aligned}\overline{F}^{\max}_{1,0}(\theta) &= \overline{F}^{\max}_{1,1}(\theta,\theta_0=-\theta) = \overline{F}^{\max}_{1,1}(\theta,\theta)_0 = \pi-\theta) \\ &= \min\left\{\overline{F}^{\max}_{1,1}(\theta,\theta_0) : \theta_0 \in [-\theta,\pi-\theta]\right\}.\end{aligned} \tag{22}$$

Again

$$\overline{F}^{\max}_{1,1}\left(\theta = \frac{\pi}{2},\theta_0\right) = \frac{1}{2} + \frac{\cos\frac{\theta_0}{2}}{2\sqrt{2}},$$

which will take its maximum value $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ for $\theta_0 = 0$. Thus for estimating the direction of pure qubit, uniformly distributed on a given great circle, if two pure qubits are supplied, it is always better to supply two parallel qubits (or, equivalently two anti-parallel qubits), rather than supplying one pure qubit from the great circle and another corresponding qubit from a small circle whose plane is parallel to that of the great circle.

In the maximization procedure of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ over all values of $\theta_0 \in [-\theta,\pi-\theta]$, we therefore can assume that θ is different from 0, $\frac{\pi}{2}$ and π. In figure 7, $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ is plotted for $0 \leqslant \theta \leqslant \pi$ and $-\theta \leqslant \theta_0 \leqslant \pi - \theta$.

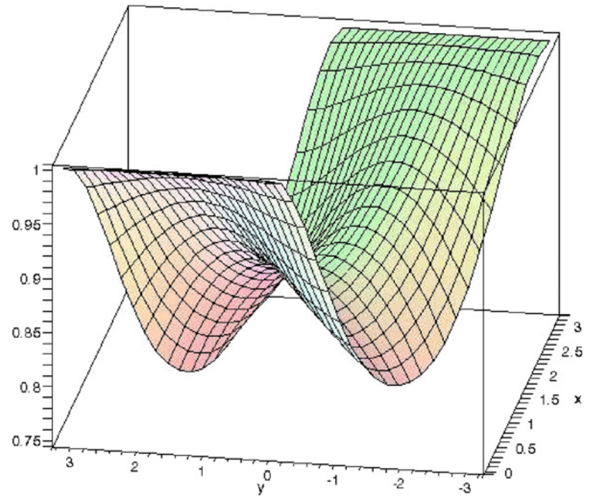

Figure 7. The plot of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ for $0 \leqslant \theta \leqslant \pi$ and $-\theta \leqslant \theta_0 \leqslant \pi-\theta$. The points of $\overline{F}^{\max}_{1,1}(\theta,0)$ are on the intersection of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ with the plane $y=0$. The points of $\overline{F}^{\max}_{1,1}(\theta,\pi-2\theta)$ are on the intersection of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ with the plane $2x+y=\pi$.

Our idea behind the choice of the set $S_{1,1}(\theta,\theta_0)$, from which a two-qubit state has to be supplied for the estimation, is to check whether for any fixed $\theta \in \left([0,\pi]-\left\{0,\frac{\pi}{2},\pi\right\}\right)$, states from $S_{1,1}(\theta,0)$ give the minimum value of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ and states from $S_{1,1}(\theta,\pi-2\theta)$ give the maximum value of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$. But one can check that for all $\theta\in[0,\pi]$,

$$\left[\frac{\partial \overline{F}^{\max}_{1,1}(\theta,\theta_0)}{\partial\theta_0}\right]_{\theta_0=0} = \frac{\sin 2\theta}{4}, \tag{23}$$

and

$$\left[\frac{\partial \overline{F}^{\max}_{1,1}(\theta,\theta_0)}{\partial\theta_0}\right]_{\theta_0=\pi-2\theta} = -\frac{1}{2}\cos^2\theta\sin 2\theta. \tag{24}$$

The right-hand sides of both (23) and (24) are equal to zero if and only if $\theta=0,\frac{\pi}{2},\pi$. Thus for given any $\theta \in \left([0,\pi]-\left\{0,\frac{\pi}{2},\pi\right\}\right)$, neither the minimum value of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ is attained by the supply of parallel qubits $\left|\Psi_{2,0}(\theta,\phi)\right\rangle$, nor the maximum value of $\overline{F}^{\max}_{1,1}(\theta,\theta_0)$ is attained by the supply of anti-parallel qubits $\left|\Psi_{1,1}(\theta,\phi)\right\rangle$. This shows that in order to extract best information about the direction of the Bloch vector $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \sin\theta)$ of a qubit $|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}|1\rangle$ (contained within the circle $S_\theta = \{|\psi(\theta,\phi)\rangle : \phi \in [0,2\pi)\}$), we need to encode the direction of the Bloch vector in a two-qubit pure state (i.e., the supplied state) in a form which in general is neither parallel nor anti-parallel.

6. Estimation of qubits from two diametrically opposite circles

We have seen that in the case of estimating the direction of the Bloch vector $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ of the qubit $|\psi(\theta,\phi)\rangle \in S_\theta = \{|\psi(\theta,\phi)\rangle : \phi\in[0,2\pi)\}$, the anti-parallel qubits $\left|\Psi_{1,1}\right\rangle = |\psi(\theta,\phi)\rangle\otimes|\psi(\pi-\theta,\pi+\phi)\rangle$ give better information compared to the parallel qubits $\left|\Psi_{2,0}\right\rangle = |\psi(\theta,\phi)\rangle\otimes|\psi(\theta,\phi)\rangle$ (where ϕ is uniformly distributed over

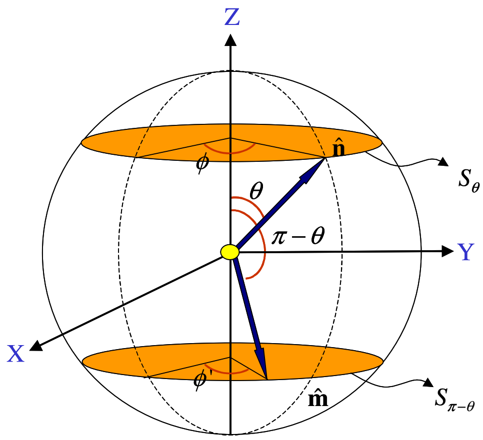

Figure 8.

$[0.2\pi)$), even though both anti-parallel as well as parallel qubits, in this scenario, span three-dimensional subspaces. By symmetry, it can be shown that in the case of estimating the direction of the Bloch vector

$$\widehat{\mathbf{m}}\,(\widehat{\mathbf{n}}) = (\sin(\pi-\theta)\cos\phi, \sin(\pi-\theta)\sin\phi, \cos(\pi-\theta)) = (\sin\theta\cos\phi, \sin\theta\sin\phi, -\cos\theta),$$

the anti-parallel qubits $|\Psi_{1,1}\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\pi-\theta,\pi+\phi)\rangle$ give better information compared to the parallel qubits $|\Psi_{2,0}\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\theta,\phi)\rangle$ (where ϕ is uniformly distributed over $[0.2\pi)$) (see figure 8). It should be noted here that the score of the game in the former case is $\frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}_r}{2}$, while, for the latter case, it is equal to $\frac{1+\widehat{\mathbf{m}}(\widehat{\mathbf{n}})\cdot\widehat{\mathbf{m}}(\widehat{\mathbf{n}}_r)}{2}$, where $\widehat{\mathbf{n}}_r = (\sin\theta\cos\phi_r, \sin\theta\sin\phi_r, \cos\theta)$. Hence, $\frac{1+\widehat{\mathbf{n}}\cdot\widehat{\mathbf{n}}_r}{2} = \frac{1+\widehat{\mathbf{m}}(\widehat{\mathbf{n}})\cdot\widehat{\mathbf{m}}(\widehat{\mathbf{n}}_r)}{2}$. Consider now the problem of estimating the direction of the Bloch vector $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ associated with the qubit $|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \in \{|\psi(\theta,\phi)\rangle : \phi \in [0, 2\pi)\}$, when the supplied two qubits can be either of the form $\left|\Psi_{2,0}(\theta,\phi)\right\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\theta,\phi)\rangle$ or of the form $\left|\Psi_{2,0}(\pi-\theta,\phi)\right\rangle = |\psi(\pi-\theta,\phi)\rangle \otimes |\psi(\pi-\theta,\phi)\rangle$, in the case of parallel qubits, while the supplied two qubits can be either of the form $\left|\Psi_{1,1}(\theta,\phi)\right\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\pi-\theta,\pi+\phi)\rangle$ or of the form $\left|\Psi_{1,1}(\pi-\theta,\phi)\right\rangle = |\psi(\pi-\theta,\phi)\rangle \otimes |\psi(\theta,\pi+\phi)\rangle$, in the case of anti-parallel qubits.

Note that the dimension of the linear spans of the sets of all parallel and anti-parallel qubits, $|\psi\rangle \otimes |\psi\rangle$ and $|\psi\rangle \otimes |\psi^\perp\rangle$, are respectively *three* and *four*, whenever the qubit $|\psi\rangle$ is taken from the set of two diametrically opposite circles

$$S_{\theta,\pi-\theta} = \{|\psi(\theta,\phi)\rangle : \phi \in [0, 2\pi)\} \cup \{|\psi(\pi-\theta,\phi)\rangle : \phi \in [0, 2\pi)\}, \tag{25}$$

where $\theta \in [0,\pi] - \left\{0, \frac{\pi}{2}, \pi\right\}$ is arbitrary but fixed. (The bases of the spans are in fact $\{|00\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\}$ and $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle + |10\rangle\}$, respectively.) So, according to dimensional argument, anti-parallel qubits should give more information about the direction of the Bloch vector of the qubit $|\psi\rangle$, compared to parallel qubits. The question is here to choose an appropriate score. The general estimation strategy is then as follows. For any qubit $|\psi\rangle = |\psi(\theta',\phi')\rangle$ (where $\theta' \in [0,\pi]$ and $\phi' \in [0, 2\pi)$), we denote by $|\psi^\perp\rangle$ the corresponding orthogonal qubit $|\psi(\pi-\theta',\pi+\phi')\rangle$. Let

$$S^{\parallel}_{\theta,\pi-\theta} = \{|\psi\rangle \otimes |\psi\rangle : |\psi\rangle \in S_{\theta,\pi-\theta}\} \qquad \text{and} \qquad S^{\perp}_{\theta,\pi-\theta} = \{|\psi\rangle \otimes |\psi^\perp\rangle : |\psi\rangle \in S_{\theta,\pi-\theta}\}, \tag{26}$$

where $S_{\theta,\pi-\theta}$ is given in (25). If a state $|\Psi^{\|}\rangle \equiv |\psi\rangle \otimes |\psi\rangle$ is supplied from $S^{\|}_{\theta,\pi-\theta}$, we perform a POVM $\mathcal{M} = \{A_r = C_r P[\alpha_{1r}|00\rangle + \alpha_{2r}|11\rangle + \alpha_{3r}|\psi^+\rangle] : r \in \Lambda\}$ on this state (where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$); if the rth outcome of the measurement occurs (with probability $\langle\Psi^{\|}|A_r|\Psi^{\|}\rangle$), the estimated qubit is taken as the density matrix ρ_r (and hence, the score is $s(\mathcal{M}, \mathcal{T} = \{\rho_r : r \in \Lambda\}, |\psi\rangle) = \langle\psi|\,\rho_r\,|\psi\rangle$). On the other hand, if a state $|\Psi^{\perp}\rangle \equiv |\psi\rangle \otimes |\psi^{\perp}\rangle$ is supplied from $S^{\perp}_{\theta,\pi-\theta}$, we perform a POVM $\mathcal{M} = \{A_r = C_r P\,[\alpha_{1r}|00\rangle + \alpha_{2r}|11\rangle + \alpha_{3r}|01\rangle + \alpha_{4r}|10\rangle] : r \in \Lambda\}$ on this state (where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$); and if the rth outcome of the measurement occurs (with probability $\langle\Psi^{\perp}|A_r|\Psi^{\perp}\rangle$), the estimated qubit is taken as the density matrix ρ_r (and hence, the score is $s(\mathcal{M}, \mathcal{T} = \{\rho_r : r \in \Lambda\}, |\psi\rangle) = \langle\psi|\,\rho_r\,|\psi\rangle$). Thus the average fidelity of estimation for parallel and anti-parallel qubits are respectively given by

$$\begin{aligned}\overline{F}_{\theta,\pi-\theta;\|}(\mathcal{M}, \mathcal{T}) &= \frac{1}{2} \times \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \left\{ \sum_{r\in\Lambda} \langle\Psi_{2,0}(\theta,\phi)|A_r|\Psi_{2,0}(\theta,\phi)\rangle \langle\psi(\theta,\phi)|\rho_r|\psi(\theta,\phi)\rangle \right\} \mathrm{d}\phi \\ &\quad + \frac{1}{2} \times \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \left\{ \sum_{r\in\Lambda} \langle\Psi_{2,0}(\pi-\theta,\phi')|A_r|\Psi_{2,0}(\pi-\theta,\phi')\rangle \right. \\ &\quad \left. \times \langle\psi(\pi-\theta,\phi')|\rho_r|\psi(\pi-\theta,\phi')\rangle \right\} \mathrm{d}\phi', \end{aligned} \tag{27}$$

and

$$\begin{aligned}\overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M}, \mathcal{T}) &= \frac{1}{2} \times \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \left\{ \sum_{r\in\Lambda} \langle\Psi_{1,1}(\theta,\phi)|A_r|\Psi_{1,1}(\theta,\phi)\rangle \langle\psi(\theta,\phi)|\rho_r|\psi(\theta,\phi)\rangle \right\} \mathrm{d}\phi \\ &\quad + \frac{1}{2} \times \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \left\{ \sum_{r\in\Lambda} \langle\Psi_{1,1}(\pi-\theta,\phi')|A_r|\Psi_{1,1}(\pi-\theta,\phi')\rangle \right. \\ &\quad \left. \times \langle\psi(\pi-\theta,\phi')|\rho_r|\psi(\pi-\theta,\phi')\rangle \right\} \mathrm{d}\phi'. \end{aligned} \tag{28}$$

Since our motivation is to estimate the direction of the Bloch vector of the qubit taken from $S_{\theta,\pi-\theta}$, the estimated qubit ρ_r should be of the form

$$\rho_r = \lambda_r|\psi(\theta,\phi_r)\rangle\langle\psi(\theta,\phi_r)| + (1-\lambda_r)|\psi(\pi-\theta,\phi_r')\rangle\langle\psi(\pi-\theta,\phi_r')|, \tag{29}$$

where $0 \leqslant \lambda_r \leqslant 1, \phi_r, \phi_r' \in [0, 2\pi)$. The parameters $\lambda_r, \phi_r, \phi_r'$ need to be chosen in such a way that average state estimation fidelities would become maximum for the given POVM $\mathcal{M}$. For our purpose, we take $\lambda_r = 1$. The reason behind this choice is the following: the optimal state estimation fidelity for estimating the direction of the Bloch vector $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ of the qubit $|\psi(\theta,\phi)\rangle \in S_\theta = \{|\psi(\theta,\phi)\rangle : \phi \in [0, 2\pi)\}$, when the supplied state is $|\Psi_{2,0}(\theta,\phi)\rangle$, is the same as for estimating the direction of the Bloch vector $\widehat{\mathbf{m}}(\widehat{\mathbf{n}}) = (\sin\theta\cos\phi, \sin\theta\sin\phi, -\cos\theta)$, even when the supplied state is as above. This is also true for anti-parallel states. It follows that the maximum values of

$$\frac{1}{2\pi} \int_{\phi=0}^{2\pi} \sum_{r\in\Lambda} \langle\Psi_{j,k}(\theta,\phi)|A_r|\Psi_{j,k}(\theta,\phi)\rangle |\langle\psi(\theta,\phi)|\psi(\theta,\phi_r)\rangle|^2 \,\mathrm{d}\phi$$

and

$$\frac{1}{2\pi} \int_{\phi=0}^{2\pi} \sum_{r\in\Lambda} \langle\Psi_{j,k}(\theta,\phi)|A_r|\Psi_{j,k}(\theta,\phi)\rangle |\langle\psi(\pi-\theta,\phi')|\psi(\pi-\theta,\phi_r)\rangle|^2 \,\mathrm{d}\phi',$$

where $(j, k) \in \{(2, 0), (1, 1)\}$, are equal.

6.1. Parallel case

With the choice of the estimated state $\rho_r = P[|\psi(\theta, \phi_r)\rangle]$, for the rth measurement outcome of the POVM

$$\mathcal{M} = \{E_r = C_r P[\alpha_{1r}|00\rangle + \alpha_{2r}|11\rangle + \alpha_{3r}|\psi^+\rangle] : r \in \Lambda\},$$

the average fidelity when parallel qubits are supplied is

$$\overline{F}_{\theta,\pi-\theta;\parallel}(\mathcal{M}, \mathcal{T}) = \frac{1}{2}\left(1 + \frac{(2-\cos\theta)\sin^3\theta}{4\sqrt{2}}\sum_{r\in\Lambda} C_r|\alpha_{1r}\alpha_{3r}|\cos(\varepsilon_{1r} - \varepsilon_{3r} + \phi_r)\right.$$
$$\left. + \frac{(2+\cos\theta)\sin^3\theta}{4\sqrt{2}}\sum_{r\in\Lambda} C_r|\alpha_{2r}\alpha_{3r}|\cos(\varepsilon_{2r} - \varepsilon_{3r} - \phi_r)\right), \qquad (30)$$

where $\alpha_{jr} = |\alpha_{jr}|\,\mathrm{e}^{\mathrm{i}\varepsilon_{jr}}$ for $j = 1, 2, 3$ and $r \in \Lambda$. We have to maximize $\overline{F}_{\theta,\pi-\theta;\parallel}(\mathcal{M}, \mathcal{T})$ over all possible choices of $\mathcal{M}$ and $\mathcal{T}$, and subject to the constraints (A), (B), and (C), that is,

$$\begin{aligned} &C_r > 0 && \text{for all} \quad r \in \Lambda, \\ &\sum_{j=1}^{3} |\alpha_{jr}|^2 = 1 && \text{for all} \quad r \in \Lambda \\ &\sum_{r\in\Lambda} C_r \alpha_{jr}\alpha_{kr}^* = \delta_{jk} && \text{for all} \quad j, k = 1, 2, 3. \end{aligned} \qquad (31)$$

From (30) it follows that

$$\overline{F}_{\theta,\pi-\theta;\parallel}(\mathcal{M}, \mathcal{T}) \leqslant \frac{1}{2}\left[1 + \frac{(2-\cos\theta)\sin^3\theta}{4\sqrt{2}}\left(\sum_{r\in\Lambda} C_r|\alpha_{1r}|^2\right)^{1/2}\left(\sum_{r\in\Lambda} C_r|\alpha_{3r}|^2\right)^{1/2}\right.$$
$$\left. + \frac{(2+\cos\theta)\sin^3\theta}{4\sqrt{2}}\left(\sum_{r\in\Lambda} C_r|\alpha_{2r}|^2\right)^{1/2}\left(\sum_{r\in\Lambda} C_r|\alpha_{3r}|^2\right)^{1/2}\right]$$
$$= \frac{1}{2}\left(1 + \frac{(2-\cos\theta)\sin^3\theta}{4\sqrt{2}} + \frac{(2+\cos\theta)\sin^3\theta}{4\sqrt{2}}\right)$$
$$= \frac{1}{2}\left(1 + \frac{\sin^3\theta}{\sqrt{2}}\right). \qquad (32)$$

Let us choose

$$\begin{aligned} &\Lambda = \{1, 2, 3\}, \\ &C_r = 1 \qquad \text{for all} \quad r \in \Lambda, \\ &\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{31} = \alpha_{13} = \frac{1}{\sqrt{3}}, \quad \alpha_{22} = \alpha_{33} = \frac{e^{4\pi \mathrm{i}/3}}{\sqrt{3}}, \quad \alpha_{32} = \alpha_{23} = \frac{\mathrm{e}^{2\pi \mathrm{i}/3}}{\sqrt{3}}, \\ &\phi_1 = 0, \quad \phi_2 = \frac{2\pi}{3}, \quad \phi_3 = \frac{4\pi}{3}. \end{aligned} \qquad (33)$$

For the choice (33), one can see that all the conditions in (31) are satisfied, then

$$\overline{F}_{\theta,\pi-\theta;\parallel}(\mathcal{M}, \mathcal{T}) = \frac{1}{2}\left(1 + \frac{\sin^3\theta}{\sqrt{2}}\right) = \overline{F}^{\max}_{\theta,\pi-\theta;\parallel}. \qquad (34)$$

The elements of the POVM can be expressed in the following matrix in terms of the basis $\{|00\rangle, |11\rangle, |\psi^+\rangle\}$

$$\begin{bmatrix} \alpha_{11} = \frac{1}{\sqrt{3}} & \alpha_{21} = \frac{1}{\sqrt{3}} & \alpha_{31} = \frac{1}{\sqrt{3}} \\ \alpha_{12} = \frac{1}{\sqrt{3}} & \alpha_{22} = \frac{e^{4\pi i/3}}{\sqrt{3}} & \alpha_{32} = \frac{e^{2\pi i/3}}{\sqrt{3}} \\ \alpha_{13} = \frac{1}{\sqrt{3}} & \alpha_{23} = \frac{e^{2\pi i/3}}{\sqrt{3}} & \alpha_{33} = \frac{e^{4\pi i/3}}{\sqrt{3}} \end{bmatrix}.$$

This matrix is the discrete Fourier transform of dimension 3.

6.2. *Anti-parallel case*

With the choice of the estimated state $\rho_r = P[|\psi(\theta, \phi_r)\rangle]$, for the rth measurement outcome of the POVM

$$\mathcal{M} = \{E_r = C_r P[\alpha_{1r}|00\rangle + \alpha_{2r}|11\rangle + \alpha_{3r}|01\rangle + \alpha_{4r}|10\rangle] : r \in \Lambda\},$$

the average fidelity when anti-parallel qubits are supplied is

$$\begin{aligned} \overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M}, \mathcal{T}) = \frac{1}{2}\Bigg[1 &- \frac{\sin^3\theta}{4} \sum_{r\in\Lambda} C_r |\alpha_{1r}\alpha_{3r}| \cos(\varepsilon_{1r} - \varepsilon_{3r} + \phi_r) \\ &+ \frac{\sin^3\theta}{4} \sum_{r\in\Lambda} C_r |\alpha_{1r}\alpha_{4r}| \cos(\varepsilon_{1r} - \varepsilon_{4r} + \phi_r) \\ &+ \frac{\sin^3\theta}{4} \sum_{r\in\Lambda} C_r |\alpha_{2r}\alpha_{3r}| \cos(\varepsilon_{2r} - \varepsilon_{3r} - \phi_r) \\ &- \frac{\sin^3\theta}{4} \sum_{r\in\Lambda} C_r |\alpha_{2r}\alpha_{4r}| \cos(\varepsilon_{2r} - \varepsilon_{4r} - \phi_r) \Bigg], \end{aligned} \tag{35}$$

where $\alpha_{jr} = |\alpha_{jr}|\,e^{i\varepsilon_{jr}}$ for $j = 1, 2, 3, 4$ and $r \in \Lambda$. We have to maximize $\overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M}, \mathcal{T})$ over all possible choices of $\mathcal{M}$ and $\mathcal{T}$, and subject to the constraints (A), (B), and (C), that is,

$$\begin{aligned} & C_r > 0 && \text{for all} \quad r \in \Lambda, \\ & \sum_{j=1}^{4} |\alpha_{jr}|^2 = 1 && \text{for all} \quad r \in \Lambda, \\ & \sum_{r\in\Lambda} C_r \alpha_{jr}\alpha_{kr}^* = \delta_{jk} && \text{for all} \quad j, k = 1, 2, 3, 4. \end{aligned} \tag{36}$$

From (35) it follows that

$$\overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M}, \mathcal{T}) = \frac{1}{2} + \frac{\sin^3\theta}{8} \sum_{r\in\Lambda} C_r \,\mathrm{Re}[(\alpha_{3r}^* - \alpha_{4r}^*)(\alpha_{2r}\,e^{-i\phi_r} - \alpha_{1r}\,e^{i\phi_r})]. \tag{37}$$

Then

$$\begin{aligned} \overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M}, \mathcal{T}) &\leqslant \frac{1}{2} + \frac{\sin^3\theta}{8} \sum_{r\in\Lambda} C_r |\alpha_{3r}^* - \alpha_{4r}^*| \times |\alpha_{2r}\,e^{-i\phi_r} - \alpha_{1r}\,e^{i\phi_r}| \\ &\leqslant \frac{1}{2} + \frac{\sin^3\theta}{8} \left[\left(\sum_{r\in\Lambda} C_r |\alpha_{3r}^* - \alpha_{4r}^*|^2 \right) \left(\sum_{r\in\Lambda} C_r |\alpha_{2r}^* - \alpha_{1r}^*|^2 \right) \right]^{1/2} \end{aligned}$$

$$= \frac{1}{2} + \frac{\sin^3\theta}{8}\left[\sum_{r\in\Lambda} C_r|\alpha_{3r}|^2 + \sum_{r\in\Lambda} C_r|\alpha_{4r}|^2 - 2\,\mathrm{Re}\left(\sum_{r\in\Lambda} C_r\alpha_{3r}\alpha_{4r}^*\right)\right]^{1/2}$$
$$\times\left[\sum_{r\in\Lambda} C_r|\alpha_{1r}|^2 + \sum_{r\in\Lambda} C_r|\alpha_{2r}|^2 - 2\,\mathrm{Re}\left(\sum_{r\in\Lambda} C_r\alpha_{2r}\alpha_{1r}^*\,\mathrm{e}^{-2i\phi_r}\right)\right]^{1/2}$$
$$= \frac{1}{2} + \frac{\sin^3\theta}{4}\left[1 - \mathrm{Re}\left(\sum_{r\in\Lambda} C_r\alpha_{1r}\alpha_{2r}^*\,\mathrm{e}^{2i\phi_r}\right)\right]^{1/2}$$
$$\leqslant \frac{1}{2}\left(1 + \frac{\sin^3\theta}{\sqrt{2}}\right). \tag{38}$$

Let us choose

$$\begin{aligned}
&\Lambda = \{1,2,3,4\},\\
&C_r = 1 \qquad \text{for all} \quad r\in\Lambda,\\
&\alpha_{1r} = \frac{\mathrm{e}^{-2i\phi_r}}{2}, \qquad \alpha_{2r} = -\frac{1}{2} \qquad \text{for all} \quad r\in\Lambda,\\
&\alpha_{3r} = \frac{-\mathrm{e}^{-i\phi_r}}{\sqrt{2}} \qquad \text{for} \quad r=1,2 \quad \text{and} \quad \alpha_{3r}=0 \quad \text{for} \quad r=3,4,\\
&\alpha_{4r} = 0 \qquad \text{for} \quad r=1,2 \quad \text{and} \quad \alpha_{4r} = \frac{\mathrm{e}^{-i\phi_r}}{}\sqrt{2} \quad \text{for} \quad r=3,4,\\
&\phi_1 = \frac{\pi}{2}, \quad \phi_2 = \frac{3\pi}{2}, \quad \phi_3 = 0 \quad \text{and} \quad \phi_4 = \pi.
\end{aligned} \tag{39}$$

For the choice (39), one can see that all the conditions in (36) are satisfied, then

$$\overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M},\mathcal{T}) = \frac{1}{2}\left(1 + \frac{\sin^3\theta}{\sqrt{2}}\right) = \overline{F}^{\max}_{\theta,\pi-\theta;\perp} = \overline{F}^{\max}_{\theta,\pi-\theta;\parallel}. \tag{40}$$

Expressed in terms of the basis $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$, the elements of the POVM give the following matrix

$$\begin{bmatrix}
\alpha_{11} = -\frac{1}{2} & \alpha_{21} = -\frac{1}{2} & \alpha_{31} = \frac{\mathrm{i}\sqrt{2}}{2} & \alpha_{41} = 0\\
\alpha_{12} = -\frac{1}{2} & \alpha_{22} = -\frac{1}{2} & \alpha_{32} = -\frac{\mathrm{i}\sqrt{2}}{2} & \alpha_{42} = 0\\
\alpha_{13} = \frac{1}{2} & \alpha_{23} = -\frac{1}{2} & \alpha_{33} = 0 & \alpha_{43} = \frac{1}{\sqrt{2}}\\
\alpha_{14} = \frac{1}{2} & \alpha_{24} = -\frac{1}{2} & \alpha_{34} = 0 & \alpha_{44} = -\frac{1}{\sqrt{2}}
\end{bmatrix}. \tag{41}$$

If instead of this strategy we use a POVM whose elements (expressed in terms of the above basis) give the Fourier transform of dimension 4, we can find that

$$\overline{F}_{\theta,\pi-\theta;\perp}(\mathcal{M},\mathcal{T}) = \frac{1}{2}\left[1 + \frac{\sin^3\theta}{16}\left(2\sqrt{2}\sin\left(\frac{\pi}{4} - \phi_2\right) - 4\cos\phi_3 + 2\sin\phi_4\right)\right],$$

which attains the maximum value $\frac{1}{2} + \left(\frac{(3+\sqrt{2})\sin^3\theta}{8}\right)$, for $\phi_2 = \frac{7\pi}{4}, \phi_3 = \pi, \phi_4 = \frac{\pi}{2}$ and ϕ_1 arbitrary. Observe that $\frac{1}{2} + \left(\frac{(3+\sqrt{2})\sin^3\theta}{8}\right) < \frac{1}{2}\left(1 + \frac{\sin^3\theta}{\sqrt{2}}\right)$, the value in (40). Finally, it is important to remark that even if

$$\dim\mathcal{L}\left(S^{\parallel}_{\theta,\pi-\theta}\right) = 3 < \dim\mathcal{L}\left(S^{\perp}_{\theta,\pi-\theta}\right) = 4,$$

we have that $\overline{F}^{\max}_{\theta,\pi-\theta;\perp} = \overline{F}^{\max}_{\theta,\pi-\theta;\parallel}$. So, once again, the dimensional argument fails. Observe that the matrix (41) is similar to

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

the Haar transform of dimension 4 (see, e.g. [6]).

7. LOCC protocol

We describe here an LOCC protocol for optimally estimating the direction of a qubit $|\psi(\theta,\phi)\rangle$ (θ is fixed), when the supplied states are parallel states $|\Psi_{2,0}\rangle = |\psi(\theta,\phi)\rangle \otimes |\psi(\theta,\phi)\rangle$. For LOCC protocols the optimality does not change whether the supplied two qubits are parallel or anti-parallel (or anything else), as far they are product states. The steps of the protocol are the following:

1. Perform the PV (projection valued) measurement $\left\{P\left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right], P\left[\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right]\right\}$ on the first qubit.

2.1. If $P\left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right]$ is the outcome of the measurement in (1) (the probability of this event being $\frac{1}{2}(1+\sin\theta\cos\phi)$), the PV measurement $\left\{P\left[\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{i}|1\rangle)\right], P\left[\frac{1}{\sqrt{2}}(|0\rangle-\mathrm{i}|1\rangle)\right]\right\}$ is performed on the second qubit.

2.1.1. If $P\left[\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{i}|1\rangle)\right]$ is the outcome of the measurement in (2.1) (with probability $\frac{1}{2}(1+\sin\theta\sin\phi)$), the estimated state is taken as $|\psi(\theta,\frac{\pi}{4})\rangle$.

2.1.2. If $P\left[\frac{1}{\sqrt{2}}(|0\rangle-\mathrm{i}|1\rangle)\right]$ is the outcome of the measurement in (2.1) (with probability $\frac{1}{2}(1-\sin\theta\sin\phi)$), the estimated state is taken as $\left|\psi\left(\theta,\frac{7\pi}{4}\right)\right\rangle$.

2.2. If $P\left[\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right]$ is the outcome of the measurement in (1) (the probability of this event being $\frac{1}{2}(1-\sin\theta\cos\phi)$), the PV measurement $\left\{P\left[\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{i}|1\rangle)\right], P\left[\frac{1}{\sqrt{2}}(|0\rangle-\mathrm{i}|1\rangle)\right]\right\}$ is performed on the second qubit.

2.2.1. If $P\left[\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{i}|1\rangle)\right]$ is the outcome of the measurement in (2.2) (with probability $\frac{1}{2}(1+\sin\theta\sin\phi)$), the estimated state is taken as $\left|\psi\left(\theta,\frac{3\pi}{4}\right)\right\rangle$.

2.2.2. If $P\left[\frac{1}{\sqrt{2}}(|0\rangle-\mathrm{i}|1\rangle)\right]$ is the outcome of the measurement in (2.2) (with probability $\frac{1}{2}(1-\sin\theta\sin\phi)$), the estimated state is taken as $\left|\psi\left(\theta,\frac{5\pi}{4}\right)\right\rangle$.

The average fidelity is then given by

$$\begin{aligned}\overline{F}_{2,\mathrm{LOCC}}(\theta) = \frac{1}{2\pi}\int_{\phi=0}^{2\pi} \Bigg\{ &\left[\frac{1}{2}(1+\sin\theta\cos\phi)\times\frac{1}{2}(1+\sin\theta\sin\phi)\times\left|\left\langle\psi\left(\theta,\frac{\pi}{4}\right)\middle|\psi(\theta,\phi)\right\rangle\right|^2\right] \\ &+\left[\frac{1}{2}(1+\sin\theta\cos\phi)\times\frac{1}{2}(1-\sin\theta\sin\phi)\times\left|\left\langle\psi\left(\theta,\frac{7\pi}{4}\right)\middle|\psi(\theta,\phi)\right\rangle\right|^2\right] \\ &+\left[\frac{1}{2}(1-\sin\theta\cos\phi)\times\frac{1}{2}(1+\sin\theta\sin\phi)\times\left|\left\langle\psi\left(\theta,\frac{3\pi}{4}\right)\middle|\psi(\theta,\phi)\right\rangle\right|^2\right] \\ &+\left[\frac{1}{2}(1-\sin\theta\cos\phi)\times\frac{1}{2}(1-\sin\theta\sin\phi)\times\left|\left\langle\psi\left(\theta,\frac{5\pi}{4}\right)\middle|\psi(\theta,\phi)\right\rangle\right|^2\right]\Bigg\}\,\mathrm{d}\phi\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8\pi}\int_{\phi=0}^{2\pi}\left\{\left[1+\sqrt{2}\sin\theta\sin\left(\phi+\frac{\pi}{4}\right)+\frac{\sin^2\theta\sin 2\phi}{2}\right]\right.\\
&\quad\times\left[1-\sin^2\theta\sin^2\left(\frac{\phi}{2}-\frac{\pi}{8}\right)\right]\\
&\quad+\left[1-\sqrt{2}\sin\theta\sin\left(\phi-\frac{\pi}{4}\right)-\frac{\sin^2\theta\sin 2\phi}{2}\right]\times\left[1-\sin^2\theta\sin^2\left(\frac{\phi}{2}-\frac{7\pi}{8}\right)\right]\\
&\quad+\left[1+\sqrt{2}\sin\theta\sin\left(\phi-\frac{\pi}{4}\right)-\frac{\sin^2\theta\sin 2\phi}{2}\right]\times\left[1-\sin^2\theta\sin^2\left(\frac{\phi}{2}-\frac{3\pi}{8}\right)\right]\\
&\quad+\left[1-\sqrt{2}\sin\theta\sin\left(\phi+\frac{\pi}{4}\right)+\frac{\sin^2\theta\sin 2\phi}{2}\right]\\
&\quad\left.\times\left[1-\sin^2\theta\sin^2\left(\frac{\phi}{2}-\frac{5\pi}{8}\right)\right]\right\}\mathrm{d}\phi\\
&= \frac{1+\cos^2\theta}{2}+\frac{\sin^3\theta}{2\sqrt{2}}.
\end{aligned}
$$

The value obtained is then equal to $\overline{F}_{2,0}^{\max}(\theta)$ (where all types of measurements are allowed). Thus

$$\overline{F}_{2,\mathrm{LOCC}}^{\max}(\theta) = \overline{F}_{2,0}^{\max}(\theta) = \frac{1+\cos^2\theta}{2}+\frac{\sin^3\theta}{2\sqrt{2}}, \qquad \text{for every} \quad \theta\in[0,\pi].$$

The optimal fidelity obtained by performing LOCC is then equal to the optimal fidelity obtained by performing joint measurements on parallel qubits.

In the attempt to generalize the above analysis to N parallel qubits, there is some evidence that von Neumann measurements on individual qubits may not achieve the optimal fidelity. A POVM (on the Nth qubit) consisting of 2^{N-2} rank-1 elements may in fact give better fidelity [19].

8. Discussion

Gisin and Popescu [10] have shown that more information about the direction of (the Bloch vector of) a qubit $|\psi(\theta,\phi)\rangle = \cos(\theta/2)|0\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)|1\rangle$ can be obtained from anti-parallel states $|\Psi_{1,1}(\theta,\phi)\rangle = |\psi(\theta,\phi)\rangle\otimes|\psi(\pi+\theta,\pi-\phi)\rangle$, compared to parallel states $|\Psi_{2,0}(\theta,\phi)\rangle = |\psi(\theta,\phi)\rangle\otimes|\psi(\theta,\phi)\rangle$, where (θ,ϕ) is uniformly distributed over $[0,\pi]\times[0,2\pi)$. They attributed the cause of this fact to the difference between the dimensions of the subspaces spanned by parallel and anti-parallel states, respectively.

When $\theta=\pi/2$, there is no difference in the amount of information as in that case (and only in that case) exact spin flipping is possible. For any fixed θ, the dimension of the space spanned by N and M qubits respectively identical and orthogonal to $|\psi(\theta,\phi)\rangle$ is $(N+M+1)$. We found that, whenever we fix $\theta\neq 0, \neq\pi/2$ or $\neq\pi$, anti-parallel states always give more information about the direction of the qubit. We generalized this to the case of N and M qubits respectively identical and orthogonal to $|\psi(\theta,\phi)\rangle$. Here the measurement basis for the optimal estimation strategy always turns out to be the Fourier basis.

We considered the case of two diametrically opposite circles. We found that both the sets $\{|\Psi_{1,1}(\theta,\phi)\rangle : \phi\in[0,2\pi)\}\cup\{|\Psi_{1,1}(\pi-\theta,\pi+\phi)\rangle : \phi\in[0,2\pi)\}$ and $\{|\Psi_{2,0}(\theta,\phi)\rangle : \phi\in[0,2\pi)\}\cup\{|\Psi_{2,0}(\pi-\theta,\pi+\phi)\rangle : \phi\in[0,2\pi)\}$ give the same information about the direction of $|\psi(\theta,\phi)\rangle$, even though the linear span of the first set is four and the one of the second set is three. The scenario described is not exactly phase estimation, still the

Fourier basis is again the optimal measurement basis for the case of parallel qubits. This does not hold for anti-parallel qubits.

We have seen that encoding of the two qubits to parallel or anti-parallel states is nothing special in regard to optimal extraction of information about the direction of the qubit $|\psi(\theta, \phi)\rangle$ from a fixed circle. In particular we have seen that the encoding of $|\psi(\theta, \phi)\rangle$ in the form $|\psi(\theta, \phi)\rangle \otimes |\psi(\theta + \theta_0, \phi)\rangle$, where θ_0 is fixed in $[-\theta, \pi - \theta]$, can provide more information compared to the case of anti-parallel qubits, when $\theta_0 \neq 0, \neq \pi/2$ or $\neq \pi$.

When two parallel qubits are supplied from a circle, we have verified that a measurement strategy using LOCC can give rise to optimal information. This is interesting from the experimental point of view since it is practically difficult to perform measurements in an entangled basis (see, e.g. [4]).

Massar [13] has shown that in the case of extracting information about the direction of a qubit taken from a uniform distribution over the whole Bloch sphere, even parallel qubits can give better information compared to anti-parallel qubits provided one chooses the proper score. From the point of view of estimation of statistical parameters, this argument is of course reasonable. This does not shed light on the reason whether there is some physical connection between impossibility of spin flipping and outperformance of anti-parallel over parallel qubits. Moreover it is not clear what kind of score is preferable, given some supplied multiqubit states; even though the physically motivated score should be the one which directly estimates the direction of the qubit. (We considered this score.)

We conclude with some open problems:

- Determine which one of the following two sets provides more information about the direction of $|\psi(\theta, \phi)\rangle$: $\{|\Psi_{n,N-n}(\theta, \phi)\rangle : \phi \in [0, 2\pi)\} \cup \{|\Psi_{n,N-n}(\pi - \theta, \pi + \phi)\rangle : \phi \in [0, 2\pi)\}$ and $\{|\Psi_{N,0}(\theta, \phi)\rangle : \phi \in [0, 2\pi)\} \cup \{|\Psi_{N,0}(\pi - \theta, \pi + \phi)\rangle : \phi \in [0, 2\pi)\}$.
- Given any $\theta \in [0, \pi]$, determine for which values of $\theta_0 \in [-\theta, \pi - \theta]$, which one of the following sets provides more information about the direction of $|\psi(\theta, \phi)\rangle$: $\{|\psi(\theta, \phi)\rangle^{\otimes n} \otimes |\psi(\theta + \theta_0, \phi)\rangle^{\otimes (N-n)} : \phi \in [0, 2\pi)\}$.
- Given $\theta \in [0, \pi]$, determine whether a strategy using LOCC is optimal for estimating the direction of $|\psi(\theta, \phi)\rangle$, when the supplied state belongs to the set $\{|\Psi_{N,0}(\theta, \phi)\rangle : \phi \in [0, 2\pi)\}$. If this is true, what is then the corresponding LOCC?

Acknowledgments

SLB currently holds a Royal Society-Wolfson Research Merit award. This work is funded in part by EPSRC grants GR/87406 and GR/S56252. SG would like to thank Emili Bagan and Ramon Muñoz-Tapia for useful discussion regarding this paper and for indicating [2]. Part of this work has been done while the authors were at The Isaac Newton Institute for Mathematical Sciences (Cambridge, United Kingdom), during the Quantum Information Science Programme (16 Aug–17 Dec 2004).

References

[1] Davies E B 1978 Information and quantum measurement *IEEE Trans. Inform. Theory* **24** 596–9
Akhiezer N I and Glazman I M 1981 *Theory of Linear Operators in Hilbert Space* vol I (translated from the third Russian edition by E. R. Dawson; translation edited by W N Everitt) (*Monographs and Studies in Mathematics* vol 9) (*Advanced Publishing Program*) (Boston, MA: Pitman)

[2] Bagan E, Baig M, Brey A, Muñoz-Tapia R and Tarrach R 2000 Optimal strategies for sending information through a quantum channell *Phys. Rev. Lett.* **24** 5230
Bagan E, Baig M, Brey A, Muñoz-Tapia R and Tarrach R 2001 Optimal encoding and decoding of a spin direction *Phys. Rev.* A **63** 052309

[3] Bagan E, Monras A and Muñoz-Tapia R 2004 Estimation of pure qubits with collective and individual measurements *Preprint* quant-ph/0412027
[4] Bouwmeester D, Ekert A and Zeilinger A (ed) 2000 *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation* (Berlin: Springer)
[5] D'Ariano G M, Macchiavello C and Sacchi M F 1998 On the general problem of quantum phase estimation *Phys. Lett.* A **248** 103
[6] Daubechies I C and Gilbert A C 1999 Harmonic analysis, wavelets and applications *Hyperbolic Equations and Frequency Interactions (Park City, UT, 1995) (IAS/Park City Math. Ser.* vol 5*)* (Providence, RI: American Mathematical Society) pp 159–226
[7] Derka R, Bužek V and Ekert A K 1998 Universal algorithm for optimal estimation of quantum states from finite ensembles via realizable generalized measurement *Phys. Rev. Lett.* **80** 1571
[8] Ghosh S, Roy A and Sen U 2001 Antiparallel spin does not always contain more information *Phys. Rev.* A **63** 014301
[9] Gill R and Massar S 2000 State estimation for large ensembles *Phys. Rev.* A **61** 042312
[10] Gisin N and Popescu S 1999 Spin flips and quantum information for antiparallel spins *Phys. Rev. Lett.* **83** 432
[11] Holevo A S 2001 *Statistical Structure of Quantum Theory (Lecture Notes in Physics, Monographs* vol 67*)* (Berlin: Springer)
[12] Latorre J I, Pascual P and Tarrach R 1999 Minimal optimal generalized quantum measurements *Phys. Rev. Lett.* **81** 1351
[13] Massar S 2000 Collective versus local measurements on two parallel or antiparallel spins *Phys. Rev.* A **62** 040101
[14] Massar S and Popescu S 1995 Optimal extraction of information from finite quantum ensembles *Phys. Rev. Lett.* **74** 1259
[15] Pati A K and Braunstein S L 2000 Impossibility of deleting an unknown quantum state *Nature* **404** 164–5
[16] Peres A 1993 *Quantum Theory: Concepts and Methods (Fundamental Theories of Physics* vol 57*)* (Dordrecht: Kluwer)
[17] Peres A and Wootters W K 1999 Optimal detection of quantum information *Phys. Rev. Lett.* **66** 1119
[18] Wootters W K and Zurek W H 1982 A single quantum cannot be cloned *Nature* **299** 802–3
[19] Work in progress